The Schrödinger, Heisenberg and Interaction pictures

These are three different ways of describing the time development of a physical system. We will derived the Heisenberg Picture (HP) and the Interaction picture (IP) from the Schrödinger Picture (SP).

In the SP the time dependence is part of state describing the physical system.

$$i\hbar \frac{d}{dt} |A,t\rangle_s = H |A,t\rangle_s .$$

To emphasise it is the <u>state</u> that changes with time in the Schrödinger picture.

$$\underset{\uparrow}{|A,t\rangle_S}$$
function of time.

We know that in the S.P. the wave equation can be solved in terms of the wave function at an arbitrary starting time t_0.

$$|A,t\rangle_s = U |A,t_0\rangle_S$$

$\uparrow$ $U \equiv U(t,t_0) = e^{-iH(t-t_0)/\hbar}$

By writing the time dependence explicitly for the S.P. we can transfer this dependence from the state to the operator. In this way we can change from the S.P. to the H.P.

So let us define

$$|A,t\rangle_H = U^{\dagger}|A,t\rangle_S = |A,t_0\rangle_S$$

Thus the in H.P. the state does <u>not</u> vary with time.

Since

$$\langle A,t|_S O_S |A,t\rangle_S = \langle A,t|_H O_H |A,t\rangle_H$$

as they both measure $\langle O\rangle$ (the mean value of the operator O) the same in both pictures.

Using the substitution above

$$\langle A,t|_S O_S |A,t\rangle_S = \langle A,t|_S U O_H U^{\dagger}|A,t\rangle_S$$

$$\Rightarrow \quad O_S = U O_H U^{\dagger}$$

$$\Rightarrow \quad \boxed{O_H = U^{\dagger} O_S U}$$

From the above we can see that

O_H is now a function of t through U^{+} and U, but $|A,t\rangle_H = |A,t_0\rangle_S$ which are no longer functions of time.

Note that from

$$O^H(t) = U^{+} O^S U$$

$$H^H(t) = U^{+} H^S U$$

$$= e^{+i(t-t_0)H_s} H_s e^{-i(t-t_0)H_s}$$

But since $e^{i(t-t_0)H_s}$ is simply a series in powers of H_s we can move the H_s through the series and write

$$H^H(t) = H^S e^{i(t-t_0)H_s} e^{-i(t-t_0)H}$$

$$\therefore H^H(t) = H^S$$

Which it had to be because H is independent of time so $H^H(t)$ had to just be H^S.

Note the unitary transformation connecting the S.P and the H.P. not only ensures the correspondence between (4)

$$ {}_S\langle B, t | O^S | A, t \rangle_S = {}_H\langle B, t | O^H | A, t \rangle_H $$

but also ensures that if

$$ [O^S, P^S] = \text{Const} = [O^H, P^H]. $$

<u>Proof</u>

$$
\begin{aligned}
[O^S, P^S] &= O^S P^S - P^S O^S = \text{const} \\
&= U^\dagger O^S U U^\dagger P^S U - U^\dagger P^S U U^\dagger O^S U = \text{const} U^\dagger U \\
&= O^H(t), P^H(t) - P^H(t) \overset{H}{O}(t) = \text{const} \\
&\Rightarrow [O^H(t), P^H(t)] = \text{const}
\end{aligned}
$$

Note also that by differentiation of

$$O^H(t) = U^+ O^S U$$

we get the Heisenberg equation of motion.

since $U^+ = e^{iH(t-t_0)/\hbar}$

$$\frac{dU^+}{dt} = \frac{i}{\hbar} H e^{iH(t-t_0)/\hbar} = \frac{i}{\hbar} H U^+$$

so

$$i\hbar \frac{d}{dt} O^H(t) = i\hbar\left[\frac{dU^+}{dt} O^S U + U^+ O^S \frac{dU}{dt}\right]$$

$$= i\hbar\left[\frac{i}{\hbar} H U^+ O^S U + U^+ O^S U H \frac{-i}{\hbar}\right]$$

$$= i\hbar\left[\frac{i}{\hbar}\left[H O^H(t) - O^H(t) H\right]\right]$$

$$= O^H(t) H - H O^H(t)$$

$$= \left[O^H(t), H\right]$$

$$\therefore \quad i\hbar \dot{O}^H(t) = \left[O^H(t), H\right]$$

Heisenberg's Equation of motion.

N.B. If an S.P. operator were time dependent the derivative on the previous page would have an extra term.

$$U^{+} \, i\hbar \frac{dO^{S}}{dt} \, U$$

$$= i\hbar \frac{\partial}{\partial t} U^{+} O^{S} U$$

$$= i\hbar \frac{\partial}{\partial t} O^{H}(t) \rightarrow$$

where $\frac{\partial}{\partial t}$ implies it is only operating on the O^{S} part CONFUSING!

The Interaction Picture

The I.P. arises if the Hamiltonian is split into two parts.

$$H = H_0 + H_I$$

H_I – This is the part of the Hamiltonian that describes the interaction between two field (or more)

H_0 – is the part of the Hamiltonian that describes the "free" fields that are interacting in H_I.

Let us define the unitary operator

$$U_0 \equiv U_0(t, t_0) = e^{-iH_0(t-t_0)/\hbar}$$

only the "free" part of Hamiltonian

An now define an interaction picture state wave function as

$$|A, t\rangle_I = U_0^{+} |A, t\rangle_S$$

note difference $H \to U^{+}$s.

Thus we can also define the interaction operator

$$O^I(t) = U_0^{+} O^S U_0$$

Differentiating the equation above (using same method] gives

$$i\hbar \frac{d}{dt} O^I(t) = [O^I(t), H_0]$$

So $O^I(t)$ commutes only with H_0 ("free" Hamiltonian)

So substituting

$$|A,t\rangle_I = U_0^+ |A,t\rangle_S$$

$$\Rightarrow \quad |A,t\rangle_S = U_0 |A,t\rangle_I$$

into Schrodinger Equation:

$$i\hbar \frac{d}{dt} |A,t\rangle_S = H |A,t\rangle_S$$

gives

$$i\hbar \frac{d}{dt} U_0 |A,t\rangle_I = H\, U_0 |A,t\rangle_I$$

Differentiating the l.h.s.

$$i\hbar\left(\frac{-i}{\hbar}\right)H_0U_0|A,t\rangle_I + i\hbar\, U_0\frac{d|A,t\rangle_I}{dt} = H\,U_0|A,t\rangle_I$$

$$\therefore\ i\hbar\, U_0\frac{d}{dt}|A,t\rangle_I = (H-H_0)U_0|A,t\rangle_I$$

Thus multiplying both sides by U_0^{+} $\qquad U_0^{+}U_0 = \mathbb{1}$

$$i\hbar\frac{d}{dt}|A,t\rangle_I = \underbrace{U_0^{+}\underbrace{(H-H_0)}_{H_I}U_0}_{H_I^{I}}|A,t\rangle_I$$

Thus

$$\boxed{i\hbar\frac{d}{dt}|A,t\rangle_I = H_I^{I}|A,t\rangle_I}$$

Thus the interaction picture wave functions defined above where $H_I^{I} = U_0^{+}H_IU_0$

$$|A,t\rangle_I \equiv U_0^{+}|A,t\rangle_S$$

satisfies the "interaction" Hamiltonian above which only considers the interaction part of the Hamiltonian.

INTERACTION HEISENBERG SCHRODINGER PICTURES

SUMMARY

HEISENBERG

$$\boxed{i\hbar \frac{d}{dt} |A,t\rangle_S = H |A,t\rangle_S}$$

$$\boxed{|A,t\rangle_S = U|A,t_0\rangle_S = e^{-iH(t-t_0)} |A,t_0\rangle}$$

$$\boxed{U \equiv e^{-iH(t-t_0)}}$$

define Heisenberg time independent

$$\boxed{|A,t_0\rangle_S = U^+ |A,t\rangle_S \equiv |A,t\rangle_H}$$

$$\boxed{O_H = U^+ O_S U}$$

⇐ NB H already time independent

Same Eigenvalues:

$$\boxed{[O_S, H_S] = \text{const} = [O_H, H_H]}$$

Heisenberg Equations Motion

$$\boxed{O^H(t) = U^+ O^S U}$$

Differentiate both sides

$$\boxed{i\hbar \dot{O}^H(t) = [O^H(t), H]}$$

$$\boxed{H = H_0 + H_I}$$

Define
$$\boxed{\begin{aligned} U_0 &\equiv U_0(t\, t_0) = e^{-iH_0(t-t_0)} \\ U &\equiv e^{-iH(t-t_0)} \end{aligned}}$$

Define

$$\boxed{|A,t\rangle_I = U_0^{+}\, |A,t\rangle_S}$$

so

$$\boxed{O^{I}(t) = U_0^{+}\, O^{S}\, U_0}$$

as before differentiating

$$\boxed{i\hbar \frac{d}{dt} O^{I}(t) = [O^{I}(t), H_0]}$$

commutes only with H_0

substituting into schroedinger and differentiating

$$\boxed{i\hbar \frac{d}{dt} |A,t\rangle_I = H^{I}\, |A,t\rangle_I}$$