Causality

The state

$\phi(x)|0\rangle$

Let's take an hermitian Klein-Gordon field $\phi(x)$ (ie real) as the simplest example.

So that we can define the probability of finding a particle at a give space time point $x = (t, \underline{x})$ one needs to define a state where at time x^0 a particle is localized at position $\underline{x}$.

If this were the case the inner product of $\phi(t, \underline{x})|0\rangle$ with $\phi(t, y)|0\rangle$ show be zero unless $\underline{x} = \underline{y}$ at equal times.

We will see this is not the case.

We, for convenience, divide ϕ into the positive-energy part and the negative energy part.

$$\phi(x) = \phi_a(x) + \phi_a^+(x)$$

where

$$\phi_a(x) \equiv \sum_{p} a_p e_p(x)$$

$$\phi_a^+(x) \equiv \sum_{p} a_p^+ e_p^*(x).$$

since $a_p|0\rangle = 0$ and $\langle 0|a_p^+ = 0$.

Thus the inner product of $\phi(x)|0\rangle$ and $\phi(y)|0\rangle$ (using hermiticity) $(\phi^+ = \phi)$

$$\langle 0|\phi^+(x)\ \phi(y)|0\rangle$$

$$= \langle 0|\ \phi(x)\ \phi(y)\ |0\rangle \quad \text{— hermiticity.}$$

$$= \langle 0|\ \phi_a(x)\ \phi_a^+(y)\ |0\rangle$$

since this is only combination that will not annihilate the vacuum.

$$= \langle 0 | \sum_{\underline{p}} a_{\underline{p}} e_{\underline{p}}(x) \sum_{\underline{p}'} a^{+}_{\underline{p}'} e_{\underline{p}'}(x) | 0 \rangle$$

$$= \langle 0 | \sum_{\underline{p}, \underline{p}'} \underbrace{a_{\underline{p}} a^{+}_{\underline{p}'}}_{\delta_{\underline{p}\underline{p}'} + \underbrace{a^{+}_{\underline{p}'} a_{\underline{p}}}_{0}} e_{\underline{p}}(x) e^{*}_{\underline{p}'}(y) | 0 \rangle$$

$$= \langle 0 | 0 \rangle \sum e_{\underline{p}}(x) e^{*}_{\underline{p}}(y)$$

This expression can be written as an integral over continuous p space, each $\left(\frac{V}{(2\pi)^3}\right)^{-1}$ representing one state of $\underline{p}$. $\therefore \sum_{p} = \int d^3p / (V/(2\pi)^3)$.

$$\therefore \sum_{p} e_{\underline{p}}(x) e^{*}_{\underline{p}}(y) = \frac{V}{(2\pi)^3} \int d^3p \, \frac{e^{-i\underline{p}\cdot\underline{x}}}{\sqrt{2p^0 V}} \frac{e^{i\underline{p}\cdot\underline{y}}}{\sqrt{2p^0 V}}$$

$$= \frac{1}{(2\pi)^3} \int \frac{d^3p}{2p^0} e^{-ip\cdot(x-y)}$$

$$\boxed{\text{Let} \quad \Delta_{+}(x-y) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2p^0} e^{-ip\cdot(x-y)}}$$

Define

$$\Delta_+(z) \equiv \frac{1}{(2\pi)^3}\int \frac{d^3p}{2p^0}\, e^{-ip^\mu z_\mu}$$

where z is a real 4-vector and $p^0 \equiv \sqrt{\underline{p}^2+M^2}$

Thus in terms of this new delta function we have

$$\boxed{\langle 0|\,\phi(x)\,\phi(y)\,|0\rangle = \Delta_+(x-y)}$$

So far x and y are arbitrary. Now assume $x^0 = y^0$ (equal time). Thus the time component of z becomes zero

$$\Delta_+(\underline{z}, 0) = \frac{1}{(2\pi)^3}\int \frac{d^3p}{2p^0}\, e^{i\underline{p}.\underline{z}}$$

$$= \frac{1}{(2\pi)^3}\int \frac{2\pi\, p^2\, dp\, d\cos\theta}{2\sqrt{p^2+m^2}}\, e^{ipr\cos\theta}$$

where $p = |\underline{p}|$ $\quad r = |\underline{z}|$.

This integral is expressible in terms of of the modified Bessel function $K_1(z)$.

K is a solution of the differential equation

$$z^2 X'' + z X' - (z^2 + n^2) X = 0.$$

with $n = 1$:

$$\Delta_+(0, \underline{z}) = \frac{M}{4\pi^2 r} K_1(mr) \overset{r \to \infty}{\sim} \frac{\sqrt{M}}{(2\pi r)^{3/2}} e^{-Mr}.$$

while this is a rapidly decreasing function of r it is not zero!

$$\boxed{\text{Thus the state } \phi(t, \underline{x})\,|0\rangle \text{ cannot be interpreted as a state which is entirely located at } x.}$$

It is of course sharply peaked there but there is a leakage away from that point.

$$(X^0 > 0) \quad \Delta_+(x) = \begin{cases} \dfrac{m}{8\pi\sqrt{x^2}}\left[Y_1(m\sqrt{x^2}) + i J_1(m\sqrt{x^2})\right] & (x^2 > 0) \\ -\dfrac{m}{4\pi^2\sqrt{-x^2}} K_1(m\sqrt{-x^2}) & (x^2 < 0). \end{cases}$$

$(x^0 < 0)$: use $\Delta_+(-x) = \Delta_+^*(x)$ - follow from defin to Δ_+.

Where J_1 and Y_1 are the standard bessel functions that are two solution of

$$z^2 X'' + z X' + (z^2 - n^2) X = 0 \quad (n = 1).$$

The function $\Delta_+(x)$ is real and positive for $x^2 < 0$.

and complex and oscillatory for $x^2 > 0$.

Lets look at this again in greater depth.

First note that the |norm of $\phi(x)|0\rangle$ is infinity.

$$\langle 0|\, \phi^{\dagger}(x)\ \phi(x)\,|0\rangle = \Delta_{+}(0)$$

$$= \int \frac{d^3p}{(2\pi)^3 2p^0} = \infty$$

Since we are in the Heisenberg picture the states are constant and the operator carries all the time dependence.

The state $\phi(t,\underline{x})|0\rangle$ represents a particle nearly localized at $(t,\underline{x})$, but through the time varying operators it also represents the entire history of the evolution from the infinite past to the infinite future.

So what is the proper interpretation of the inner product of the two states

$\phi(x)|0\rangle$ and $\phi(y)|0\rangle$?

The simplest interpretation is in the Schrodinger picture.

For simplicity set $y^0 = 0$ and $x^0 > y^0$.

At $t = y^0 = 0$ the states and the operators in the two pictures (Schrodinger and Heisenberg) are identical.

In the Schrodinger picture the operators are

$$\phi_s(\underline{x}) \equiv \phi(0,\underline{x}) \qquad \phi_s(\underline{y}) \equiv \phi(0,\underline{y})$$

$$|\underline{x}\rangle_s \equiv \phi_s(\underline{x})|0\rangle \qquad |\underline{y}\rangle_s \equiv \phi_s(\underline{y})|0\rangle$$

where 's' is for Schrodinger.

The time evolution of the state $|\underline{y}\rangle_s$ is given by $|t,\underline{y}\rangle_s = e^{-iHt}|\underline{y}\rangle_s$.

Now using the following :-

If $F(\phi(x), \pi(x))$ is an arbitary function (namely polynomial) of ϕ and $\pi \equiv \dot{\phi}$ then

$$e^{iP.a}F(x)e^{-iP.a} = F(x+a)$$

provided ϕ satisfies the equations of motion where the operator $F(x)$ is a function of $x = (t, \underline{x})$ through $\phi(x)$ and $\pi(x)$ and a^μ is a real * 4-vector constant.

$$\phi(x^0, \underline{x}) = e^{iHx^0} \underbrace{\phi(0, \underline{x})}_{\phi_s(\underline{x})} e^{-iHx^0}$$

Then the inner product $\langle 0 | \phi^+(x) \phi(y) | 0 \rangle$ with $y^0 = 0$ can be written as

$$\langle 0 | \phi^+(x^0, \underline{x}) \phi(y^0, \underline{y}) | 0 \rangle$$

$$= \underbrace{\langle 0 | e^{iHx^0}}_{\langle 0 |} \phi_s^+(\underline{x}) \underbrace{e^{-iHx^0} \overbrace{\phi(0, \underline{y}) | 0 \rangle}^{|0, \underline{y}\rangle_s}}_{|x^0, \underline{y}\rangle_s}$$

$$= {}_s\langle \underline{x} | x^0, \underline{y} \rangle_s .$$

where we have used $H = 0$ for the vacuum (H is normal ordered), which leads to

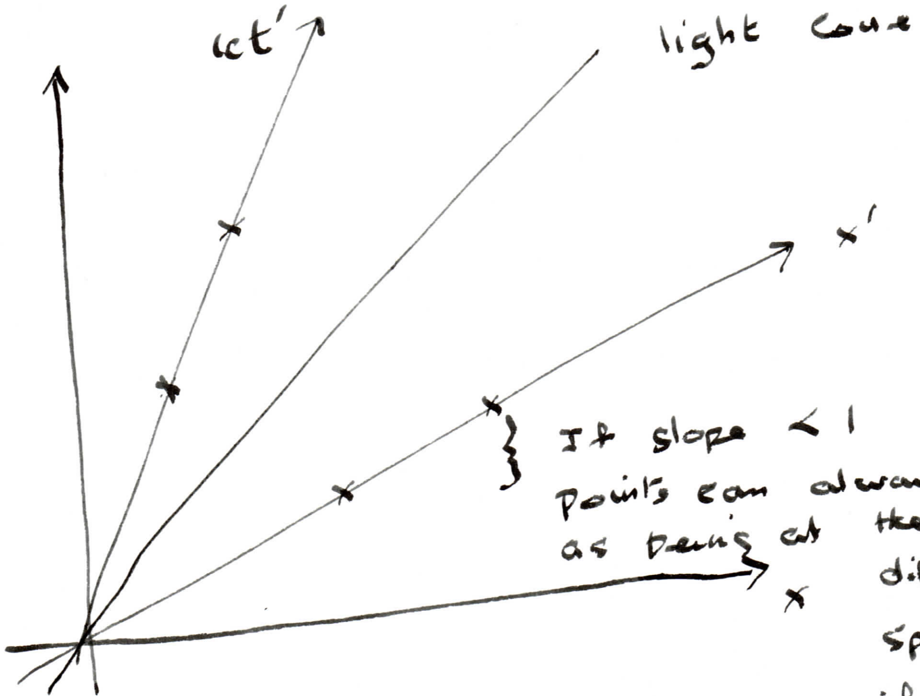
$$e^{-iHx^0} |0\rangle_s = |0\rangle_s$$

* I will derive this in an appendix.

If we interpret the state $|0, \underline{y}\rangle_s$ as the creation of a particle at position $\underline{y}$ at $t=0$, then the state $|x^0, y\rangle_s$ is its time-evolved state at time x_0. The inner product above would then be the amplitude for finding the particle at position $(\underline{x})$.

Thus $\langle 0|\phi^+(x)\,\phi(y)\,|0\rangle$ can be loosely interpreted as the amplitude for a particle created at $(y^0, \underline{y})$ to propagate to $(x^0, \underline{x})$.

So what do we make of the fact that the inner product is non-zero even if the separation between x and y is space like.



If slope < 1 These points can always be described as being at the same time different point spacially. Clearly if these are space like separation in this frame the points cannot be causally connected.

any signal would have ∞ velocity.

Space - Time translation: Appendix A

$$[A, BC] = ABC - BCA$$

$$[A, B]C = ABC - BAC.$$

$$B[A, C] = BAC - BCA.$$

$$\therefore [A, BC] = [A, B]C + B[A, C]$$

$$= B[A, C] + [A, B]C$$

Now if $[A, B^{n-1}B] = B^{n-1}[A, B] + [A, B^{n-1}]B$

but $[A, B^{n-2}B]B = B^{n-2}[A, B]B + [A, B^{n-2}]B^2$

This will carry on until $n - j = 0$

with final term $+ \quad [A, B]B^{n-1}$

Now if 6)

$$
\begin{aligned}
[A, B^n C^m] = \; & \underbrace{[A, B^n]} C^m + B^n \underbrace{[A, C^m]} \\
& B^{n-1}[A,B]\, C^m \quad + B^n C^{m-1}[A,C] \\
& + B^{n-2}[A,B] B C^m + B^n C^{m-2}[A,C] C \\
& \ldots \qquad \ldots \\
& + [A,B] B^{n-1} C^m + B^n [A,C] C^{m-1}
\end{aligned}
$$

c)

if $P^\mu = -\int d^3x' [\pi(x') \nabla' \phi(x')]$

$$[P^\mu, \phi(x)] = -i\partial^\mu \phi(x)$$

$$[P^\mu, \pi(x)] = -i\partial^\mu \pi(x)$$

$$[P^\mu, \sum_{n,m=0}^{\infty} C_{n,m} \phi^n \pi^m]$$

$$= \sum_{n,m=0}^{\infty} [P^\mu, \phi^n \pi^m]$$

since $[A, BC] = B[A,C] + [A,B]C$.

$$= \phi^n [P^\mu, \pi^m] + [P^\mu, \phi^n] \pi^m$$

$$= \phi^n \pi^{m-1} [P^\mu, \pi] + \phi^{n-1} [P^\mu, \phi] \pi^m$$

$$+ \phi^n \pi^{m-2} [P^\mu, \pi] \pi + \phi^{n-2} [P^\mu, \phi] \phi \pi^m$$

$$\cdots \qquad \cdots$$

$$+ \phi^n [P^\mu, \pi] \pi^{m-1} + [P^\mu, \phi] \phi^{n-1} \pi^m$$

which if $[P^\mu, \pi] \pi^{m-1}$ etc commute

$$= + m\phi^n (i\partial^\mu \pi) \pi^{m-1} + n\phi^{n-1} i\partial^\mu \phi \pi^m$$

So if $\quad P^\mu = -\int d^3x' \left[\pi(x')\nabla'\phi(x')\right]$ d)

we have shown

$$[\underline{P}, \phi(x)] = -\int d^3x' \left[\pi(x')\overbrace{\nabla'\phi(x'), \phi(\underline{x})}^{\substack{\nabla\phi,\,\phi \text{ commute.} \\ = 0}}\right]$$

$$= -\underbrace{\int d^3x' \left[\pi(\underline{x}'), \phi(x)\right]}_{-i\delta^3(\underline{x}'-\underline{x})} \nabla'\phi(x')$$

$$= i\nabla\phi(\underline{x})$$

$$[\underline{P}, \pi(x)] = -\int d^3x' \left[\pi(\underline{x}')\underline{\nabla}'\phi(\underline{x}'), \pi(\underline{x})\right]$$

$$= -\int d^3x' \left(\pi(\underline{x}')\left[\nabla'\phi(x'), \pi(\underline{x})\right]\right)$$

$$= i\,\underline{\nabla}\pi(\underline{x})$$

Together with

$$i\dot{\phi} = [P^0, \phi] \qquad i\dot{\pi} = [P^0, \pi]$$

$$\therefore \boxed{\begin{aligned} [P^\mu, \phi(x)] &= -i\partial^\mu\phi(x) \\ [P^\mu, \pi(x)] &= -i\partial^\mu\pi(x) \end{aligned}}$$

Now if $$F(\phi,\pi) = \sum_{n,m=0}^{\infty} c_{n,m}\, \phi^n \pi^m$$ e)

(ie an arbitrary polynomial)

$$[P^\mu, F(\phi,\pi)] = -i\partial^\mu F(\phi,\pi)$$

which we shall prove using the prior information

$$\left[P^\mu, \sum_{n,m=0}^{\infty} c_{n,m}\, \phi^n \pi^m\right]$$

$$= \sum_{n,m=0}^{\infty} c_{n,m} [P^\mu, \phi^n \pi^m]$$

now using the relation

$[A, B^n C^n]$

$$= \sum_{n,m=0}^{\infty} c_{n,m} \Big\{ \phi^{n-1}[P^\mu,\phi]\pi^m + \phi^n \pi^{m-1}[P^\mu,\pi]$$

$$+ \phi^{n-2}[P^\mu,\phi]\phi\,\pi^m + \phi^n \pi^{m-2}[P^\mu,\pi]\pi$$

$$\cdots \qquad \cdots$$

$$[P^\mu,\phi]\phi^{n-1}\pi^m + \phi^n [P^\mu,\pi]\pi^{m-1}$$

where $[P^\mu,\phi] = i\partial^\mu\phi$ $\quad [P^\mu,\pi] = i\partial^\mu\pi$

and note $[i\partial^\mu\phi, \phi] = 0$ $\quad [i\partial^\mu\pi, \pi] = 0.$

Thus

$$= \sum_{n,m=0}^{\infty} c_{n,m} \; \phi^{n-1}\, i\partial^\mu\phi\, \pi^m \quad + \phi^n\, \pi^{m-1}\, i\partial^\mu \pi$$

$$+ \phi^{n-2}\, i\partial^\mu\phi\, \phi\, \pi^m \quad + \phi^n\, \pi^{m-2}\, i\partial^\mu\pi\, \pi$$

$$\ldots \qquad \ldots$$

$$+ i\partial^\mu\phi\, \phi^{n-1}\, \pi^m \quad + \phi^n\, i\partial^\mu\pi\, \pi^{m-1}$$

using $[i\partial^\mu\phi, \phi] = 0$

$$= \sum_{n,m=0}^{\infty} c_{nm} \; n\, \phi^{n-1}\, i\partial^\mu\phi\, \pi^m \quad + \phi^n m\, \pi^{m-1}\, i\partial^\mu\pi$$

$$= \sum_{n,m=0}^{\infty} c_{nm} \; i\partial^\mu \phi^n \pi^m$$

$$= i\partial^\mu \sum_{n,m=0}^{\infty} c_{nm}\, \phi^n \pi^m$$

$$= i\partial^\mu F(\phi\pi)$$

$$\boxed{[P^\mu, F(\phi,\pi)] = i\partial^\mu F(\phi,\pi).}$$

Now for a small translation $a^\mu = \epsilon^\mu$ we have to first order in ϵ^μ.

$$e^{iP^\mu \epsilon_\mu} F(x) e^{-iP^\mu \epsilon_\mu}$$

$$= (1 + iP^\mu \epsilon_\mu) F (1 - iP^\mu x_\mu)$$

$$= F + i(P^\mu \epsilon_\mu F - F P^\mu \epsilon_\mu)$$

$$= F + i \underbrace{[P^\mu, F]}_{-(\partial^\mu F)} \epsilon_\mu \, \epsilon_\mu .$$

$$= F(x) + (\partial^\mu F) \epsilon_\mu = F(x_\mu + \epsilon_\mu)$$

This is the essence of the lie algebra which generates the space-time translation

if we want a finite translation we can just apply n of these

$$\boxed{e^{iP^\mu n \epsilon_\mu} F(x) e^{-iP^\mu n \epsilon_\mu} = F(x_\mu + n\epsilon_\mu)}$$

$n \to \infty$ where $n\epsilon_\mu = a_\mu$.

$\phi(x)|0\rangle$ is not interpreted as a particle at $\underline{x}$ and t.

$\phi(x)|0\rangle$ is interpreted as a state where a particle is created at x, then there is non-zero 'propagation' outside the light cone. This is however a violation of special relativity.

Causality

The 'propagation' outside the light cone is still a problem for special relativity. However physical events are detected through observables.

A hermitian field can be viewed as an observable.

Then causality requires that two observables $\phi(x)$ & $\phi(y)$ commute if the separation between x and y is space-like

$$\text{ie.} \quad [\phi(\underline{x}), \phi(\underline{y})] = 0 \quad \text{if } (x-y)^2 < 0$$

Note that the quantization condition states

$[\phi(t, \underline{x}), \phi(t, \underline{y})] = 0$ is an equal time commutator. Now x^0 and y^0 are in general different, so this is not the same condition.

To investigate this firstly divide ϕ into the positive and negative energy parts

$$\phi(x) = \underset{\substack{\parallel \\ \sum_p a_p e_p(\underline{x})}}{\phi_a(x)} + \underset{\substack{\parallel \\ \sum_p a_p^+ e_p^*(\underline{x})}}{\phi_a^+(x)}$$

Then the commutator can be written

$$[\phi(x), \phi(y)] = [\phi_a(\underline{x}) + \phi_a^+(\underline{x}), \phi_a(\underline{y}) + \phi^+(\underline{y})]$$

$$= [\phi_a(\underline{x})\ \phi_a^+(\underline{y})] + \underbrace{[\phi_a^+(\underline{x}), \phi_a(\underline{y})]}$$

$$- [\phi_a(y), \phi_a^+(\underline{x})]$$

$$= [\phi_a(x), \phi_a^+(y)] - (x \longleftrightarrow y).$$

$$[\phi_a(x), \phi_a^+(y)] = [\sum_p a_p e_p(\underline{x}), \sum a_{p'}^+ e_{p'}^*(\underline{y})]$$

$$= \sum e_p(x)\ e_{p'}(y)\ \underbrace{[a_p, a_{p'}^+]}_{\delta_{pp'}}$$

$$= \sum e_p(x)\ e_p(y) = \Delta_+(x-y)$$

where

$$\Delta_+(z) \equiv \frac{1}{(2\pi)^3} \int \frac{d^3p}{2p^0} \quad e^{ip\cdot z}$$

$$\text{Thus } [\phi(x), \phi(y)] = \Delta_+(z) - \Delta_+(-z) \equiv i\Delta(z)$$

since $\Delta_+(-x) = \Delta_+^*(x)$

$$\Delta_+(z) - \Delta_+(-z)$$

$$= \Delta_+(z) - \Delta_+^*(x)$$ is purely imaginary.

The factor i is added to make $\Delta(x)$ real.

Thus to have causality satisfied we must show $\Delta(z) = 0$ for $z^2 < 0$.
