Quantization of the Dirac field

The quantization of the Dirac field is the same as for K.G. but when we move into the description of the normal modes the Pauli condition must apply.

This is done by applying anti commutators among the creation and annihilation objects.

So we

1. Find the Lagrangian that gives the Dirac field
2. Derive the conjugate field and the Hamiltonian.
3. Expand the field in momentum space.
4. Apply anti commutation relations between the field and the conjugate field
5. Time variations will be given by the Heisenberg equations

The quantized system will be established.

The reason for using anticommutators was initially experimental by confirmation however we find that if the particle and antiparticles are both to have positive energy and if they satisfy causality they must have anti-commutator relations.

Lagrangian for the Dirac field:

The Lagrangian is

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi .$$

To show this consider the action principle.

$$\delta S = 0 \qquad S \equiv \int d^4x\, \mathcal{L}$$

$$= \int d^4x\, \psi^{\dagger} \gamma^{0} (i\not{\partial} - m)\psi$$

here ψ is a 4-component field with each component in principle being complex.

ψ_n and ψ_n^* $\quad n = 1, 2, 3, 4$ can be considered independent variables.

If we vary all possible variations of $\psi^{+} = (\psi_1^{*}, \psi_2^{*}, \psi_3^{*}, \psi_4^{*})$ keeping ψ unchanged.

Then

$$\delta S = \int d^4x \, \delta\psi^{+} \gamma^0 (i\not{\partial} - m)\psi = 0$$

This must hold for each of the components separately.

$$\therefore \quad \gamma^0 (i\not{\partial} - m)\psi = 0.$$

$$\Rightarrow \quad \gamma^0\gamma^0 (i\not{\partial} - m)\psi = 0.$$

$$(i\not{\partial} - m)\psi = 0.$$

which give Dirac's equation.

We will always use

$$\mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$$

The conjugate fields $(1 \to 4)$

$$\pi_n = \frac{\partial \mathcal{L}}{\partial \dot{\psi}_n} = \frac{\partial}{\partial \dot{\psi}_n}\Big[\underbrace{\bar{\psi}\gamma^\mu i\partial_\mu\psi}_{\substack{\bar{\psi}\gamma^0 i\partial_0\psi + \ldots \\ i\psi_n^* \dot{\psi}_n}} - m\bar{\psi}\psi\Big] = i\psi_n^*$$

$$\Rightarrow \boxed{\pi = i\psi^+}$$

Thus

$$\mathcal{H} = \sum_{n=1}^{4} \underbrace{\pi_n}_{i\psi_n^*}\dot{\psi}_n - \underbrace{\mathcal{L}}_{\bar{\psi}\gamma^\mu i\partial_\mu\psi - m\bar{\psi}\psi}$$

$$= i\psi^+\cancel{\dot{\psi}} - \Big(\underbrace{i\bar{\psi}\gamma^0\partial_0\psi}_{\cancel{\psi^+\dot{\psi}}} + \underbrace{i\bar{\psi}\gamma^k\partial_k\psi}_{i\psi^+\gamma^0\gamma^k\partial_k\psi} - m\psi^+\gamma^0\psi\Big)$$

thus using $\alpha_k = \gamma^0\gamma^k \quad \beta = \gamma^0$

$$\boxed{\mathcal{H} = \psi^+(-i\underline{\alpha}.\underline{\nabla} + m\beta)\psi.}$$

Note the hamiltonian

is what Dirac introduced.

$$i\partial_0\psi = (-i\underline{\alpha}.\underline{\nabla} + m\beta)\psi.$$

The total hamiltonian can be written as

$$H \equiv \int d^3x\, \mathcal{H} = \int d^3x\, \psi^\dagger i\partial_0\psi .$$

The total momentum is obtained by applying the general form.

$$p^0 = H \qquad \underline{P} = -\int d^3x\, \underbrace{\pi_\kappa}_{i\psi_n^*} \underline{\nabla}\psi_\kappa$$

$$= \int d^3x\, \psi^\dagger(-i\underline{\nabla})\psi$$

Note: For K.G.E and Dirac.E.

The operator for energy or momentum O

$O = i\partial_0 \qquad O = -i\underline{\nabla}$ is sandwiched by the appropriate inner product of the field

$\frac{1}{2}\phi(i\overleftrightarrow{\partial_0})\, O\, \phi$ — K.G.E

$\psi^\dagger O \psi$ — Dirac.

Expanding to normal modes

Suppose $\psi(t,\underline{x})$ is an arbitrary solution of the Dirac equation. Then each component at some given time eg $t=0$ $\psi_n(0,\underline{x})$ can be uniquely Fourier-expanded.

$$\psi(0,\underline{x}) = \begin{pmatrix} \psi_1(0,\underline{x}) \\ \psi_2(0,\underline{x}) \\ \psi_3(0,\underline{x}) \\ \psi_4(0,\underline{x}) \end{pmatrix} = \begin{pmatrix} \sum_{\underline{p}} c_{1\underline{p}} e^{i\underline{p}\cdot\underline{x}} \\ \sum_{\underline{p}} c_{2p} e^{i\underline{p}\cdot\underline{x}} \\ \sum_{\underline{p}} c_{3p} e^{i\underline{p}\cdot\underline{x}} \\ \sum_{p} c_{4p} e^{i\underline{p}\cdot\underline{x}} \end{pmatrix}$$

$$= \sum_{p'} \begin{pmatrix} c_{1\underline{p}} \\ c_{2p} \\ c_{3p} \\ c_{4p} \end{pmatrix} e^{i\underline{p}\cdot\underline{x}}$$

So the Dirac wave function could look like the form above but we know it is related to spinors. (PTO)

where $c_{n\underline{p}}$ $n = 1,2,3,4$ are uniquely determined complex coefficients.

For each $\underline{p}$ we can use the orthonormal set of spinors $(u_{p,\pm s}, v_{p\pm s})$ to expand

$e_{n\underline{p}}$

$$\begin{pmatrix} c_{1p} \\ c_{2p} \\ c_{3p} \\ c_{4p} \end{pmatrix} = \sum_{\underline{s}} \overbrace{A_{p,s}\, u_{p,s}}^{+\text{ energy states}} + \overbrace{B_{-p,s}\, V_{-\underline{p},s}}^{-\text{ energy states.}}$$

eg

$$u_{p+s} = \begin{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \frac{\sigma.p}{E+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix} \qquad V_{p+s} = \begin{pmatrix} \frac{\sigma.p}{E+m}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$u_{p-s} = \begin{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \frac{\sigma.p}{E+m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix} \qquad V_{p-s} = \begin{pmatrix} \frac{\sigma.\underline{p}}{E+m}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{pmatrix}$$

It can easily be shown that the normal mode functions satisfy the orthonormality conditions

$$\boxed{\begin{aligned}&\int d^3x\, f^{+}_{\underline{p}\underline{s}}(x)\, f_{\underline{p}'\underline{s}'}(x) = \int d^3x\, g^{+}_{\underline{p},\underline{s}}(x)\, g_{\underline{p}',\underline{s}'} = \delta_{pp'}\delta_{ss'}\\ &\int d^3x\, f^{+}_{\underline{p}\underline{s}}(x)\, g_{\underline{p}'\underline{s}'}(x) = \int d^3x\, g^{+}_{\underline{p},\underline{s}}(x)\, f(x)_{\underline{p}'\underline{s}'} = 0.\end{aligned}}$$

We will now regard the a's and b^{+}'s as operators in the Hilbert space and impose anticommutation relations given by

$$\boxed{\begin{aligned}&\{a_{\underline{p}\underline{s}}\ a^{+}_{\underline{p}\underline{s}}\} = \{b_{\underline{p}\underline{s}}, b^{+}_{\underline{p}'\underline{s}'}\} = \delta_{pp'}\delta_{ss'}\\ &\text{all others} = 0\end{aligned}}$$

Thus

$$\psi(0,\underline{x}) = \sum_{\underline{p},\underline{s}} (A_{p,s}\, u_{p,s} + B_{-p,s}\, V_{-p,s})\, e^{i\underline{p}\cdot\underline{x}}$$

$$= \sum_{p,s} A_{ps}\, u_{ps}\, e^{i\underline{p}\cdot\underline{x}} + \underbrace{B_{p,s}\, V_{ps}\, e^{-i\underline{p}\cdot\underline{x}}}_{p \to -p.}$$

In order that $u_{\underline{p},\underline{s}}\, e^{i\underline{p}\cdot\underline{x}}$ be associated with the positive energy frequency e^{-ip^0t} and $u_{p,s}\, e^{-i\underline{p}\cdot\underline{x}}$ the negative energy frequency e^{ip^0t} with $p^0 \equiv \sqrt{\underline{p}^2+m^2}$

$$u_{\underline{p},\underline{s}}\, e^{-i\underline{p}\cdot\underline{x}} = u_{\underline{p},\underline{s}}\, e^{-ip^0t + i\underline{p}\cdot\underline{x}}$$

$$V_{\underline{p},\underline{s}}\, e^{i\underline{p}\cdot x} = V_{\underline{p},\underline{s}}\, e^{ip^0t - i\underline{p}\cdot\underline{x}}$$

Thus the general solution can be written as.

$$\psi(t,\underline{x}) = \sum_{\underline{p},\underline{s}} A_{\underline{p},\underline{s}}\, u_{\underline{p},\underline{s}}\, e^{-ip^\mu x_\mu}$$

$$+ \quad B_{\underline{p},s}\, V_{\underline{p},\underline{s}}\, e^{ip^\mu x_\mu}.$$

Normalization

for K.G.E. the normal mode $\phi(x) = e_{\underline{p}}(x)$ was normalized so that the probability current density

$\phi^* i \overleftrightarrow{\partial} \phi$ give unity when integrated over V.

In the Dirac case the probability density is given by $j^0 = \psi^+ \psi$ thus the normalized normal mode functions are taken as

$$f_{\underline{p}\underline{s}} = \frac{u_{\underline{p}\underline{s}}}{\sqrt{2p^0 V}} e^{-i\underline{p}\cdot\underline{x}} \quad ; \quad g_{p,s} = \frac{v_{\underline{p},s}}{\sqrt{2p^0 V}} e^{i\underline{p}\cdot\underline{x}}$$

which gives

$$\int d^3x \, f^+_{\underline{p}\underline{s}}(x) \, f_{\underline{p},\underline{s}}(x) = \int_V d^3x \frac{1}{2p^0 V} \underbrace{u^+_{\underline{p},\underline{s}} u_{\underline{p},\underline{s}}}_{2p^0} = 1.$$

So the correctly normalized expansion is.

$$\boxed{\psi(x)_{t=0} = \sum_{\underline{p},\underline{s}} \left(a_{\underline{p},\underline{s}} f_{\underline{p},s}(x) + b^+_{\underline{p},\underline{s}} g_{\underline{p},\underline{s}}(x) \right)}$$

where $a_{\underline{p}\underline{s}} = \sqrt{2p^0 V} A_{\underline{p},\underline{s}}$ $\quad b^+_{\underline{p},\underline{s}} = \sqrt{2p^0 V} B_{\underline{p},\underline{s}}$

Let
$$\psi = \psi_a + \psi_{b^+}$$
$$\psi_a \equiv \sum_{\underline{p},\underline{s}} a_{\underline{p},\underline{s}} f_{\underline{p}\underline{s}}$$
$$\psi_{b^+} \equiv \sum_{\underline{p}\underline{s}} b^+_{\underline{p},\underline{s}}\, g_{\underline{p},\underline{s}}$$

Since only the $\{a a^+\}$ and $\{b b^+\}$ survive,

$$\{\psi_n(t,\underline{x}'),\ \psi_m(t,\underline{x})\}$$
$$= \{\psi^+_n(t,\underline{x}'),\ \psi^+_m(t,\underline{x})\} = 0$$

The anti commutator $\{\psi_n(t,\underline{x}),\ \psi^+_m(t,\underline{x}')\}$

$$= \{\psi_{an}(t,\underline{x}) + \psi_{b^+n}(t,\underline{x}),\ \psi^+_{am}(t,\underline{x}') + \psi^+_{b^+m}(t,\underline{x}')\}$$
$$= \underbrace{\{\psi_{an}(t,\underline{x}),\ \psi^+_{am}(t,\underline{x}')\}} + \underbrace{\{\psi_{b^+n}(t,\underline{x}),\ \psi^+_{b^+m}(t,\underline{x}')\}}$$

$$\sum_{\underline{p},\underline{s},\underline{p}',\underline{s}'} \underbrace{\{a_{\underline{p},\underline{s}},\ a^+_{\underline{p}',\underline{s}'}\}}_{\delta_{\underline{p}\underline{p}'}\,\delta_{ss'}} f_{\underline{p}\underline{s}n}(t\,\underline{x})\, f^+_{\underline{p}'\underline{s}'m}(t\,\underline{x}')$$

$$= \sum_{\underline{p}\underline{s}} f_{\underline{p}sn}(t,\underline{x}) f^+_{\underline{p}sm}(t,\underline{x}')$$

$$= \sum_{\underline{p}} \frac{e^{-i\underline{p}\cdot(\underline{x}-\underline{x}')}}{2p^0 V} \sum_{\underline{s}} (u_{\underline{p},\underline{s}}\, u^+_{p,s})_{nm}$$

$$\{\psi_{b^+}(t,x)\,,\,\psi^{+}_{b^+_m}(t,x')\}$$

$$= \underbrace{\sum_{p} \frac{e^{ip.(x-x')}}{2p^0 V}}_{\delta^3(x-x')} \underbrace{\sum_{s} (V_{-ps} V^{+}_{-p,s})_{nm}}_{\delta_{mn}}$$

given $\int d^3x\, e^{i\underline{p}\cdot\underline{x}}\, e^{-i\underline{p}'\cdot\underline{x}} = V\delta_{pp'}$

$$\left(\sum_{p} (e^{ipx})^* (e^{ipx'}) = V\delta^3(x-x')\right)$$

so $\{\psi_n(t,x)\,,\,\psi^{+}_m(t,x')\}$

$$= \delta^3(x-x')\,\delta_{mn}\,.$$

Thus using $\pi = i\psi^{+}$

$$\{\psi_n(t,\underline{x}),\ \pi_m(t,\underline{x}')\} = i\delta_{nm}\,\delta^3(\underline{x}-\underline{x}')$$

$$\{\psi_n(t,\underline{x}),\ \psi_m(t,\underline{x}')\} = \{\pi_n(t,\underline{x}),\ \pi_m(t,\underline{x}')\} = 0 .$$

Thus the a a^{+} b b^{+} anticommutation operators lead to the same equal time anticommutation reactions for the fields and conjugate Dirac fields ψ + π.

The total Energy

Recalling that

$$H = \int d^3x\, \mathcal{H} = \int d^3x\, \psi^\dagger i \partial_0 \psi$$

$$= \int d^3x \sum_{\underline{p}\,\underline{s}} \left(a^\dagger_{\underline{p}\underline{s}} f_{\underline{p},\underline{s}} + b_{\underline{p},\underline{s}}\, g^\dagger_{\underline{p},\underline{s}} \right) \sum_{\underline{p}'\underline{s}'} p^{0\prime} \left(a_{p s} f_{p s} - b^\dagger_{p's'} g_{p's'} \right)$$

$$= \sum_{p s p' s'} p^{0\prime} \left(a^\dagger_{ps}\, a_{p's'}\, \delta_{pp'}\, \delta_{ss'} - b_{ps}\, b^\dagger_{p's'}\, \delta_{\underline{p}\underline{p}'}\, \delta_{\underline{s}\underline{s}'} \right)$$

$$= \sum_{p,s} p^0 \left(a^\dagger_{\underline{p}\underline{s}}\, a_{\underline{p}\underline{s}} - b_{\underline{p}\underline{s}}\, b^\dagger_{\underline{p}\underline{s}} \right)$$

So far we have only used orthonormality conditions but the anticommutation relations allow us to write $\{ b_{ps}, b^\dagger_{ps} \} = 1$ or $b_{\underline{p}\underline{s}} b^\dagger_{\underline{p}\underline{s}} = 1 - b^\dagger_{\underline{p}\underline{s}} b_{\underline{p}\underline{s}}$

$$H = \sum_{\underline{p}\,\underline{s}} p^0 \left(a^\dagger_{ps}\, a_{ps} + b^\dagger_{ps}\, b_{ps} - 1 \right)$$

which shows that both particles + anti particles have + energy contribution.

Note however that if we had used commutation relations

$$\boxed{H = \sum_{ps} p^0 \left(a^{+}_{ps} a_{ps} - b^{+}_{ps} b_{ps} - 1 \right)}$$

COMMUTATOR RELATIONS USED

Thus antiparticles would have had — energy contribution. Note too with the commutator the energy modes can be occupied by any number of ~~states~~ states and there would have ~~been~~ been no minimum energy ~~states~~ for the vacuum.

The Lagrangian is

$$\mathcal{L} = \bar{\psi}\,(i\not{\partial} - m)\,\psi$$

$$\psi' = e^{i\theta}\psi \qquad \bar{\psi}' = e^{-i\theta}\bar{\psi}$$

$$\partial_\mu J^\mu = 0 \quad ; \quad J^\mu = i\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^+)}\phi^+ - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\phi\right)$$

$$\therefore\ j^\mu = i\left(\underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi^*_n)}}_{0}\psi^*_n - \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi_n)}}_{i(\bar{\psi}\gamma^\mu)_n}\psi_n\right)$$

$$\boxed{\therefore\ j^\mu = \bar{\psi}\,\gamma^\mu\psi.}$$

This is the probability current

The conserved quantity

$$Q = \int d^3x\, j^0 = \int d^3x\, \psi^+\psi.$$

$$\therefore \; Q \int d^3x \sum_{psp's'} \left(a^{+}_{ps} f^{+}_{ps} + b_{ps} g^{+}_{ps} \right)\left(a_{ps} f_{ps} + b^{+}_{ps} g_{ps} \right)$$

$$= \sum_{psp's'} \left(a^{+}_{ps} a_{ps} \delta_{pp'} \delta_{ss'} + b_{ps} b^{+}_{ps} \delta_{pp'} \delta_{ss'} \right)$$

$$= \sum_{ps} \left(a^{+}_{ps} a_{ps} + b_{ps} b^{+}_{ps} \right)$$

— using the anti commutation relation.

$$= \sum_{ps} \left(a^{+}_{ps} a_{ps} - b^{+}_{ps} b_{ps} + 1 \right)$$

Thus particles contribute $+1$ to Q

anti particles " -1 to Q.

but there is again the infinite sea.

$$\sum_{ps} 1$$

which can be removed by the normal ordering method : : –

$$: a^{+}_{ps} a_{ps} + b_{ps} b^{+}_{ps} : \; = a^{+}_{ps} a_{ps} - b^{+}_{ps} b_{ps}$$

namely $\quad Q \equiv \; : \int d^3x \, \psi^{+} \psi : \; = \sum_{ps} \left(a^{+}_{ps} a_{ps} - b^{+}_{ps} b_{ps} \right).$