Two Scalar fields of the same Mass

A hermitian scalar field cannot have more than one degree of freedom for a given $\underline{k}$ (ie the number of particles $N_{\underline{k}}$). This is because for a given $\underline{k}$ all we have is a pair of operators $a^+_{\underline{k}}$ $a_{\underline{k}}$.

Thus we have no more degrees of freedom to describe (eg. spin or if it is an antiparticle ie charge).

We will now consider a system with two equal mass fields with same mass in one non-hermitian field. The symmetry these two fields introduce give rise to a conservation principle which we will call charge.

We will note the correspondence

Quantum field theory	Classical Field Theory.
Hermitian field	Real field
Non-hermitian field	Complex field

Defining the Lagrangian

Consider two real fields with the same mass m, $\phi_1(x)$ and $\phi_2(x)$.

The Lagrangian density for each is given by

$$\mathcal{L}_k(\phi_k, \dot{\phi}_k) = \frac{1}{2}\left(\partial_\mu \phi_k \partial^\mu \phi_k - m\,\phi^2_k\right)$$

$k = 1, 2$.

Assuming ϕ_1 and ϕ_2 do not interact.

$$\mathcal{L}(\Phi, \dot{\Phi}) = \mathcal{L}_1(\phi_1, \dot{\phi}_1) + \mathcal{L}_2(\phi_2, \dot{\phi}_2)$$

$$= \frac{1}{2}\left[\partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi - m^2(\phi_1^2 + \phi_2^2)\right]$$

If we check the Euler-Lagrange Equations give the correct equations of motion.

$$\boxed{\underbrace{\frac{\partial \mathcal{L}}{\partial \phi_k}}_{-m^2\phi_k} = \partial_\mu \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)}}_{\partial^\mu \phi_k} \Rightarrow (\partial_\mu \partial^\mu + m^2)\phi_k = 0.}$$

The conjugate fields are by definition

$$\pi_k \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_k} = \dot{\phi}_k \quad (k = 1, 2).$$

The Hamiltonian just becomes the sum of those two systems with $\underline{\pi} = (\pi_1, \pi_2)$.

$$\mathcal{H}(\underline{\pi}, \underline{\phi}) = \sum_k \pi_k \dot{\phi}_k - \mathcal{L}$$

$$= \sum_k (\pi_k \dot{\phi}_k - \mathcal{L}_k)$$

$$= \sum_k \mathcal{H}(\pi_k, \phi_k)$$

so $H = H_1 + H_2$

with $H_k = \int d^3x \, \mathcal{H}_k(\pi_k, \phi_k)$

Let us now define two complex fields ϕ and ϕ^+ by.

$$\boxed{\begin{aligned} \phi &\equiv \frac{1}{\sqrt{2}} \left(\phi_1 + i\,\phi_2\right) \\[2ex] \phi^+ &\equiv \frac{1}{\sqrt{2}} \left(\phi_1 + i\,\phi_2\right) \end{aligned}}$$

so
$$\phi^+ \phi = \frac{1}{2} \left(\phi_1 + i\phi_2\right)\left(\phi_1 - i\phi_2\right)$$
$$= \frac{1}{2} \left(\phi_1^2 + \phi_2^2\right)$$

and
$$\partial_\mu \phi^+ \partial^\mu \phi = \frac{1}{2} \left(\partial_\mu \phi_1 - i\partial_\mu \phi_2\right)\left(\partial^\mu \phi_1 + i\partial^\mu \phi_2\right)$$
$$= \frac{1}{2}\left(\partial_\mu \phi_1 \, \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2\right)$$

So we see that the Lagrangian for the two systems can be written

$$\boxed{\mathcal{L} = \left(\partial_\mu \phi^+ \partial^\mu \phi - m^2 \phi^+ \phi\right)}$$

Since ϕ_1 and ϕ_2 both satisfy the K.G.E then ϕ and ϕ^+ (being linear sums thereof) must also be solutions.

$$(\partial^\mu + m^2)\phi = (\partial^\mu + m^2)\phi^+ = 0 .$$

Regarding ϕ and ϕ^+ as independent (rather than ϕ_1 and ϕ_2)

The Euler - Lagrange equation give

$$\underbrace{\frac{\partial \mathcal{L}}{\partial \phi}}_{-m^2\phi^+} = \partial_\mu \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}}_{\partial^\mu \phi^+} \Rightarrow (\partial_\mu \partial^\mu + m^2)\phi^+ = 0 .$$

$$\underbrace{\frac{\partial \mathcal{L}}{\partial \phi}}_{-m^2\phi} = \partial_\mu \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}}_{\partial^\mu \phi} \Rightarrow (\partial_\mu \partial^\mu + m^2)\phi = 0 .$$

The conjugate fields are

$$\pi \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}^{+} \qquad \pi^{+} \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}^{+}} = \dot{\phi}$$

or we can just differential the definition of ϕ and ϕ^{+}.

$$\pi = \frac{1}{\sqrt{2}}(\pi_1 + i\pi_2)$$

$$\pi^{+} = \frac{1}{\sqrt{2}}(\pi_1 - i\pi_2)$$

Note that this is a bit odd as we consider ϕ and ϕ^{+} as independent as clearly one is related to the other by hermitian conjugation.

How is it that all this seems to work by considering ϕ and ϕ^{+} as independent when they are clearly related?

This is the reason this procedure seems to work:—

If we regard ϕ and ϕ^+ as independent and π and π^+ as independent then the relations

$$\phi \equiv \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2) \qquad \phi^+ \equiv \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)$$

and

$$\pi = \frac{1}{\sqrt{2}}(\pi_1 + i\pi_2) \qquad \pi^+ = \frac{1}{\sqrt{2}}(\pi_1 - i\pi_2)$$

formally constitute a canonical transformation formally generated by

$$F = \frac{1}{\sqrt{2}}\left[\pi.(\phi_1 + i\phi_2) + \pi^+.(\phi_1 - i\phi_2)\right]$$

where

$$\phi = \partial F/\partial\pi \qquad \phi^+ = \partial F/\partial\pi^+$$

$$\pi_k = \frac{\partial F}{\partial\phi_k}$$

reproduce the initial relations above. Thus it is no surprise that the two 'co-ordinate systems' describe the identical dynamical system using the same $\mathcal{H}$ and $\mathcal{L}$ expressed in terms of appropriate fields.

Quantization:

This is as before. ϕ_k and π_k are operator fields and introduce equal time commutators:

$$[\phi_k(t,\underline{x}), \pi_{k'}(t,\underline{x}')] = i\delta_{kk'}\,\delta^3(x-x').$$

$$[\phi_k(t,\bar{x}), \phi_{k'}(t,\bar{x}')] = [\pi_k(t,\underline{x}), \pi_{k'}(t,\underline{x})] = 0.$$

The Heisenberg equations of motion lead to exactly the same equations of motion as before $(\partial^2+m^2)\phi_k = 0$.

Expanding in momentum modes

$$\phi_k(x) = \sum_{p} \left(a_{k,p}\, e_p(x) + a^{+}_{kp}\, e^{*}_{p}(x)\right),$$

note $e_p(x)$ is the same for both states as the p^0 is the same for the same p as m is the same. (ie no e_p k label).

So as before the commutation relations given rise to

$$[a_{kp}, a^{+}_{k'p'}]\ \delta_{\underline{k}\underline{k}'}\,\delta_{pp'}$$

$$[a_{kp}, a_{k',p'}] = [a^{+}_{k,p}, a^{+}_{k',p'}] = 0.$$

Which is the same as before.

It can easily be shown that:

$$\boxed{\begin{aligned} [\phi(t,\underline{x}), \pi(t,\underline{x}')] &= i\delta^3(\underline{x}-\underline{x}') \\ [\phi^+(t,x), \pi^+(t,x')] &= i\delta^3(\underline{x}-\underline{x}') \end{aligned}}$$

all others 0.

Note this too is what one would get if ϕ and ϕ^+ (and π and π^+) are regarded as independent fields and the standard quantization is followed.

The momentum expansion for ϕ is obtained from those of ϕ_1 and ϕ_2

$$\phi(x) \equiv \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\,\phi_2(x)\right)$$

$$= \frac{1}{\sqrt{2}}\Big[\sum_{p}\left(a_{1p}\,e_p + a_{1p}^{+}\,e_p^{*}\right)$$

$$+ i\sum_{p}\left(a_{2p}\,e_p + a_{2p}^{+}\,e_p^{*}\right)\Big]$$

$$= \sum_{p}\Big[\frac{1}{\sqrt{2}}\left(a_{1p} + i\,a_{2p}\right)e_p$$

$$+ \frac{1}{\sqrt{2}}\left(a_{1p} + i\,a_{2p}\,e_p^{*}\right)\Big]$$

Defining:

$$a_p \equiv \frac{1}{\sqrt{2}}\left(a_{1p} + i\,a_{2p}\right)$$

$$b_p \equiv \frac{1}{\sqrt{2}}\left(a_{1p} - i\,a_{2p}\right)$$

$$\Rightarrow \quad b_p^{+} = \frac{1}{\sqrt{2}}\left(a_{1p}^{+} + i\,a_{2p}^{+}\right)$$

The field ϕ can be expanded as.

$$\boxed{\phi(x) = \sum_{p} \left(a_p e_p(x) + b_p^{+} e_p^{*}(x) \right)}$$

It can easily be shown

$$[a_p, a^{+}_{p'}] = \delta_{pp'}$$

$$[b_p, b^{+}_{p'}] = \delta_{p,p'} \qquad \text{others} = 0 .$$

Thus we have annihilation operators for 2 different particles

we can also show

$$H_1 = \sum_{p} p^0 a_{1p}^{+} a_{1p}$$

$$H_2 = \sum_{p} p^0 a_{2p}^{+} a_{2p}$$

The total Hamiltonian is simply the sum.

$$H = H_1 + H_2 = \sum_{p} p^0 \left(a_{1p}^{+} a_{1p} + a_{2p}^{+} a_{2p} \right)$$

Noting that

$$a_p^{+} a_p + b_p^{+} b_p = \frac{1}{2} \left(a_{1p}^{+} - i a_{2p}^{+} \right)\left(a_{1p} + i a_{2p} \right) + \frac{1}{2} \left(a_{1p}^{+} + i a_{2p}^{+} \right)\left(a_{1p} - i a_{2p} \right)$$

$$= a_{1p}^{+} a_{1p} + a_{2p}^{+} a_{2p}$$

Therefore

$$\boxed{H = \sum_{p} p^0 \left(a_p^{+} a_p + b_p^{+} b_p \right)}$$

Thus the total energy is just the total sum of all the 'a' an 'b' particles.

Noether Symmetry and Currents:

We will show that an internal (or "isospin") symmetry has arisen because the two fields ϕ_1 and ϕ_2 have the same mass.

Let us start from non-quantized fields.

We first note the Lagrangian

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi$$

is invariant under the phase rotation transformation

$$\boxed{\phi'(x) = e^{i\theta}\phi(x) \; ; \; \phi'^\dagger = e^{-i\theta}\phi^\dagger(x)}.$$

where θ is real.

In terms of ϕ_1 and ϕ_2 the phase rotation is written as

$$\phi' \equiv \frac{1}{\sqrt{2}}(\phi_1' + i\phi_2')$$

$$= e^{i\theta}\phi = (\cos\theta + i\sin\theta)\frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$$

$$= \frac{1}{\sqrt{2}}\left[(\cos\theta\,\phi_1 - \sin\theta\,\phi_2) + i(\sin\theta\,\phi_1 + \cos\theta\,\phi_2)\right]$$

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

Thus $SO(2)$ – rotation group on ϕ_1, ϕ_2 space
(2×2)
is equivalent to $\underbrace{e^{i\theta}}\phi$
1×1 unitary matrix on ϕ.

Now considering the Lagrangian. Under space time transformation the Lagrangian density changed under its dependence on x through $\phi(x)$.

Under this transformation $\mathcal{L}$ is unchanged under the transformation.

For <u>small θ</u> the changes in the fields and their derivatives are

$$\boxed{\begin{aligned} \delta\phi &= i\theta\phi \ , \quad \delta\phi^+ = -i\theta\phi^+ \\ \delta(\partial_\mu\phi) &= i\theta\,\partial_\mu\phi, \quad \delta(\partial_\mu\phi^+) = -i\theta\,\partial_\mu\phi^+ . \end{aligned}}$$

$$\delta\mathcal{L} = \underbrace{\frac{\partial\mathcal{L}}{\partial\phi}}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi) + \underbrace{\frac{\partial\mathcal{L}}{\partial\phi^+}}\delta\phi^+ + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi^+)$$

$$\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \xleftrightarrow[\text{by Euler–Lagrange}]{} \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi^+)}$$

$$= i\theta\Big[\underbrace{\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\,\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}}_{\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\,\phi\right)} \underbrace{- \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)}\,\phi^\dagger - \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\,\partial_\mu\phi^\dagger}_{-\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\,\phi\right)} \Big]$$

$$= i\theta\,\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\,\phi - \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\,\phi^\dagger\right) = 0.$$

Thus we have a conserved current given by

$$\partial_\mu J^\mu = 0.$$

$$J^\mu = i\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi^\dagger)}\,\phi^\dagger - \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\,\phi\right)$$

The conserved quantity will be

$$\frac{dQ}{dt} = \frac{d}{dt}\int d^3x\, J^0 = 0.$$

$$\boxed{Q = \int d^3x\, J^0}$$

For the Klein - Gordon Lagrangian density

were $\mathcal{L} = \partial^\mu \phi^+ \partial_\mu \phi - m^2 \phi^+ \phi$

$$J^\mu = i\left[(\partial^\mu \phi) \phi^+ - (\partial^\mu \phi^+) \phi \right]$$

$$= \phi^+ i \overleftrightarrow{\partial^\mu} \phi$$

$$\boxed{Q = \int d^3x \, J^0 = \int d^3x \, \phi^+ i \overleftrightarrow{\partial^0} \phi.}$$

Expanding this in the momentum form.

$$Q = \int d^3x \, \phi^+ i \partial^0 \phi = \sum_p \left(a_p^+ a_p - b_p^+ b_p \right)$$

$$Q = N_a - N_b.$$

where $\quad N_a \equiv \sum_p a_p^+ a_p \qquad N_b \equiv \sum_p b_p^+ b_p$

We can use Heisenbergs Equation to verify that Q is indeed a constant of motion:

$$-i\dot{Q} = [H, Q]$$

$$= \left[\sum_p p^0 (a_p^+ a_p + b_p^+ b_p), \sum_p (a_p^+ a_p - b_p^+ b_p)\right] = 0.$$

as the number operators $N_a = a_p^+ a_p$, N_b all commute.

Thus we see that the a and b particles have an equal size and opposite sign quantum number and equal mass. This makes them good candidates to be antiparticles of each other. The quantum number is not necessarily charge at this stage as we have not defined it as being anything that couples to a photon yet.