Summary   Momentum Expansion

We can expand the K.G. wave function

as

$$\phi(t,x) = \sum c_p(t)\, \underbrace{e^{i\underline{p}\cdot\underline{x}}}_{(e^{i\underline{p}\cdot\underline{x}})^*}$$

Where $c_p = \frac{1}{V}\int d^3x \; e^{-i\underline{p}\cdot\underline{x}}\; \phi(t,x)$

as this is the standard method for projecting out a fourier series coefficient

If we substitute this in to the K.G Equation.

$$\ddot{\phi} = \nabla^2\phi - m^2\phi$$

$$\therefore \sum_p \ddot{c}_p(t)\, e^{i\underline{p}\cdot x} = \sum \left[(i p)^2 - m^2\right] c_p(t) e^{i\underline{p}\cdot\underline{x}}$$

$$\therefore \boxed{\ddot{c}_{\underline{p}}(t) = -p^{02}\, c_{\underline{p}}}$$

where we define p^0 to be a positive number $p^0 = +\sqrt{\underline{p}^2 + m^2} \geq 0$.

Thus the general solution is therefore

$$c_p(t) = A_p e^{-ip^0 t} + B_p e^{ip^0 t}$$

(without previous p^0 definition this statement would be ambiguous)

N.B. Time dependence is in the exponent A_p B_p are constants.

If ϕ is to represent an observable operator it must be hermitian

$$\therefore \phi^+ = \phi \qquad \therefore \underbrace{\sum c_p^+ e^{-i\underline{p}.\underline{x}}}_{\sum c_{-p}^+ e^{i\underline{p}.x}} = \sum c_p e^{i\underline{p}.\underline{x}}$$

$$\therefore \boxed{c_{-p}^+ = c_p}$$

If this is to be satisfied by solution above

$$A_{-p}^+ e^{ip^0 t} + B_{-p}^+ e^{-ip^0 t} = A_p e^{-ip^0 t} + B_p e^{ip^0 t}$$

Thus $\boxed{A_{-p}^+ = B_p}$

$\boxed{B_{-p}^+ = A_p}$

Substituting either one of the expressions into $\phi(t,x)$ we get

$$\phi(t,x) = \sum_{p} \left(A_p e^{-ip^\mu x_\mu} + A_p^+ e^{ip^\mu x_\mu} \right)$$

Now we define the operator a_p

$$a_p \Rightarrow A_p = \frac{1}{\sqrt{2p^0 V}} a_p .$$

We now write

$$\phi(x) = \sum_{p} a_p e_p(x) + a_p^+ e_p^*(x)$$

$$\text{where} \quad e_p(x) \equiv \frac{e^{ip\cdot x}}{\sqrt{2p^0 V}}$$

$$\pi(x) = \dot\phi(x) = \sum_{p} (-ip^0) \left(a_p e_p(x) + a_p^+ e_p^*(x) \right)$$

note a_p and a_p^+ will be shown to be creation and annihilation operators and the $e_p(x)$ are obviously just numbers.

d

we need the orthogonality relation for the $e_p(x)$.

$$\int d^3x \; e^*_p(x)\, e_{p'}(x) = \frac{1}{2V\sqrt{p^0 p^{0\prime}}} \int d^3x \; e^{ip\cdot x} e^{-ip'\cdot x}$$

$$= \frac{e^{i(p^0 - p^{0\prime})t}}{2V\sqrt{p^0 p^{0\prime}}} \int d^3x \left(e^{i\underline{p}\cdot\underline{x}}\right)^* e^{i\underline{p}'\cdot\underline{x}'}$$

$$= \frac{\delta_{pp'}}{2p^0}$$

and $$\int d^3x \; e_p(x)\, e_{p'}(x) = e^{-2ip^0 t} \frac{\delta_{p,-p'}}{2p^0}.$$

However remember that in RQM we wrote down the particle current and normalize the j^0 component

$$j^0 = \phi^* i\frac{\partial}{\partial t}\phi - i\frac{\partial}{\partial t}\phi^*\,\phi^*$$

$$= \phi^* \overleftrightarrow{\partial_0}\,\phi \equiv \phi^* i\frac{\partial}{\partial t}\phi - i\frac{\partial}{\partial t}\phi^*\,\phi$$

So the normalisation $\frac{1}{\sqrt{V 2E_0}}$ was chosen to normalize this term.

The integral $e_p(x)\, e_{p'}$ can be similarly obtained.

$$\int d^3x \; e_p(x)\, e_{p'}(x) = \frac{e^{-2ip^0 t}\, \delta_{p,-p}}{2p^0} \quad !$$

but it would have been better if it had been 0!

Now consider $a \overleftrightarrow{\partial_0} b \equiv a(\partial_0 b) - (\partial_0 a) b$.

$$\int d^3x \; e_p(x)\, i \overleftrightarrow{\partial_0}\, e_{p'}(x) = (p^{0\prime} - p^0) \underbrace{\int d^3x\, e_p(x)\, e_{p'}(x)}_{\frac{e^{-2ip^0 t}}{2p^0} \delta_{p,-p'}}$$

$$= 0$$

with $\displaystyle \int d^3x \; e_p^*(x)\, i \overleftrightarrow{\partial_0}\, e_{p'}(x) = (p^0 + p^{0\prime}) \underbrace{\int d^3x\, e_p(x)\, e_{p'}(x)}_{\frac{\delta_{p,p'}}{2p^0}}$

$$= \delta_{p,p'}$$

So the normalization for $e_p(x)$ has been chosen so that

$$\int d^3x \; e^*_p(x) i\overleftrightarrow{\partial_0} e_{p'}(x) = \delta_{pp'} \qquad \int d^3x \; e_p(x) i\overleftrightarrow{\partial_0} e^*_{p'}(x) = -\delta_{pp'}$$

and

$$\int d^3x \; e_p(x) \, i\overleftrightarrow{\partial_0} \, e_{p'}(x) = 0 \qquad \int d^3x \; e^*_p(x) \, i\overleftrightarrow{\partial_0} \, e^*_{p'}(x) = 0 .$$

Thus the orthonormality of these conditions mean we can retain the normalization for the RQM current and it also gives us the relations

$$\int d^3x \; e^*_p(x) \, i\overleftrightarrow{\partial_0} \, \phi(x) = \sum_{p} \Big[a_{p'} \underbrace{\int d^3x \; e^*_p \, i\overleftrightarrow{\partial_0} \, e_{p'}}_{\delta_{pp'}}$$

$$+ \; a^+_p \underbrace{\int d^3x \; e^*_p \, i\overleftrightarrow{\partial_0} \, e^*_{p'}}_{0} \Big] = a_p$$

writing the right hand side explicitly:

$$a_p = \int d^3x \; e^*_{p'}(x) \left[i\dot{\phi} + p^0 \phi(x) \right]$$

$$\left(\text{since } e^*_p \overleftrightarrow{\partial_0} \phi = e^*_p \dot{\phi} - p^0 e^*_p \phi \right)$$

$$\boxed{a_p = \int d^3x \; e^*_{p'}(x) \left[p^0 \phi(x) + i\pi(x) \right]}$$

Thus since ϕ and π are hermitian

$$\boxed{a^\dagger_p = \int d^3x \; e_p(x) \left[p^0 \phi(x) - i\pi(x) \right]}$$

Thus we have managed to express a_p and a_p^+ in terms of π and ϕ so we can express the commutation relations in terms of π ϕ likewise

$$\therefore \quad [a_p, a_p^+] = \Big[\int d^3x\, e_p^*(x)[p^0\phi(x) + i\pi(x)],$$

$$\int d^3x'\, e_{p'}(x)[p^0\phi(x') - i\pi(x')]\Big]$$

$$= \int d^3x \int d^3x'\, e_p^*(x) e_{p'}(x) \underbrace{[p^0\phi(x) + i\pi(x), p^{0\prime}\phi(x') - i\pi(x')]}_{i p^{0\prime}\underbrace{[\pi(x),\phi(x')]}_{-i\delta^3(\underline{x}-\underline{x}')} - i p^{0\prime}\underbrace{[\phi(x),\pi(x')]}_{i\delta^3(\underline{x}-\underline{x}')}}$$

only x terms non-zero

$$= \underbrace{\int d^3x\, e_p^*(x) e_{p'}(x)\, p^0}_{\delta_{pp'}/2p^0} + \underbrace{\int d^3x\, e_p^*(x) e_{p'}(x)\, p^{0\prime}}_{\delta_{pp'}/2p^0}$$

$$= \delta_{p,p'}$$

Thus $\quad [a_p, a_{p'}^+] = \delta_{p,p'}$

$$[a_p^+, a_{p'}^+] = [a_p, a_{p'}] = 0$$

So we can see that this is exactly what we had for the independent set of harmonic oscillators labelled by $\underline{p}$

$\therefore$ for a given $\underline{p}$ we had

$$[a \; a^{+}] = 1.$$

and the operators belonging to different oscillators commute.

Thus we can see that we can interpret

$a_{\underline{p}}$ and $a_{\underline{p}}^{+}$ as the annihilation and creation operators for the $\underline{p}$ momentum state in the wave function expansion and

$N = a_{\underline{p}}^{+} a_{\underline{p}}$ is the number of particles occupying that ~~state~~.

Total Energy and Momentum

Thus we have seen that a hermitian field that satisfies the Klein-Gordon equation can be regarded as a set of harmonic ~~operators~~ oscillators each labeled by p and with an associated pair of creation and annihilation operators a^{+} a. Then each oscillator (or normal mode) can have an integer number of quanta each being regarded as a particle with momentum p and energy p_0.

Since each normal mode is an independent oscillator there is a number oscillator for each p

$$N_p = a_p^{+} a_p$$

and a corresponding set of eigenstates

$$N_p \, |n_p\rangle_p = n_p \, |n_p\rangle_p$$

Thus just as was shown for the simple harmonic oscillator we can build any state from the ground state

$$|n_p\rangle_p = \frac{(a^{+}_p)^{n_p}}{\sqrt{n_p!}} \, |0\rangle_p$$

Let us define

$*$

$$|\{n_p\}\rangle = \prod_p |n_p\rangle_p = \left[\prod_p \frac{(a_p^+)^{n_p}}{\sqrt{n_p!}}\right]|0\rangle$$

where $|0\rangle$ is the product of all

ground states

$$|0\rangle \equiv \prod_p |0\rangle_p$$

Since each vacuum is normalized so is

the overall vacuum

$$\langle 0|0\rangle = 1 .$$

$*$ if there were just two possible momenta a and b then the total set of states described as a wave function is just

$$|n_a\rangle_a \, |n_b\rangle_b .$$

So the extension to all states is obvious.

If we allow different types of particle we just add an other label to each of the states

$$|\{n_r, p\}\rangle$$

These states with definite numbers of quanta in each particle type and momenta are called "Fock states."

For now assuming only one type of particle

Can we show

Fock STATE

$$p^\mu |\{n_p\}\rangle = \left(\sum_p n_p p^\mu\right) |\{n_p\}\rangle$$

i.e the energy and momentum is just the sum of all these existing in the universe.

we have seen that

$$\mathcal{H}(\pi, \phi, \underline{\nabla}\phi) = \frac{1}{2}(\pi^2 + (\underline{\nabla}\phi)^2 + m^2\phi^2)$$

$$H = \int d^3x \, \frac{1}{2} (\pi^2 + \underbrace{(\underline{\nabla}\phi)^2} + m^2\phi^2)$$

$$* \qquad \underbrace{\nabla \cdot (\phi \underline{\nabla}\phi)} - \phi \underbrace{\nabla^2\phi}$$

$$** \qquad \int_{\text{Surface}} \phi\nabla\phi \cdot ds = 0 \qquad \ddot{\phi} + m^2\phi \qquad \text{– using K.G.E.}$$

$$= \frac{1}{2}\int d^3x \, (\pi^2 - \phi\ddot{\phi})$$

$$= \frac{1}{2}\int d^3x \, (\dot{\phi}^2 - \phi\ddot{\phi})$$

Thus this can be written as

$$\boxed{H = \frac{1}{2}\int d^3x \, \phi \, i \overleftrightarrow{\partial_0} (i\partial_0\phi)}$$

$* \quad \nabla(\phi\nabla\phi) = \phi\nabla^2\phi + (\nabla\phi)^2$

$**$ gauss' theorem $\quad \int_{\text{volume}} \nabla \cdot A \, dv = \int_{\text{surface}} A \cdot ds$ if surface at ∞ and A finite $= 0$.

Thus

$$H = \frac{1}{2}\int d^3x \left[\sum_{p} a_p e_p + a_p^{+} e_p^{*}\right] i\partial_0 \left[\sum_{p'} p^{0\prime}(a_{p'} e_{p'} - a_{p'}^{+} e_{p'}^{*})\right]$$

$$= \frac{1}{2}\int d^3x \sum_{p p'} p^{0\prime} \Big(a_p a_{p'} \underbrace{e_p \overleftrightarrow{\partial_0} e_{p'}}_{0}$$

$$- a_p a_{p'}^{+} \underbrace{e_p\, i\overleftrightarrow{\partial_0}\, e_{p'}}_{-\delta_{pp'}}$$

$$+ a_p^{+} a_{p'} \underbrace{e_p^{*}\, i\overleftrightarrow{\partial_0}\, e_{p'}}_{\delta_{pp'}}$$

$$- a_p^{+} a_{p'}^{+} \underbrace{e_p\, i\overleftrightarrow{\partial_0}\, e_{p'}}_{0} \Big)$$

$$= \frac{1}{2}\sum_{p} p^0 \big(\underbrace{a_p a_p^{+}}_{a_p^{+} a_p + 1} + a_p^{+} a_p \big)$$

— commutation relation

$$H = \sum_{p} p^0 \left(a_p^{+} a_p + \frac{1}{2} \right)$$

This look like what needed to show that the energy of the Fock states was the sum of the individual modes & occupancy except the $\sum_{\underline{p}} p^0 \frac{1}{2}$ which is in principle infinity. The $\sum_{\underline{p}} p^0 N$ is exactly what we want however $(N \equiv a^+_{\underline{p}} a_{\underline{p}})$.

Note that $\frac{1}{2} \sum p^0$ is the energy of the state with $n_{\underline{p}} = 0$ for all $\underline{p}$, namely the energy of the vacuum $|0\rangle$. This is obvious if you just set $N = 0$ for all states.

Since we cannot ascribe an absolute value to energy only a relative value we can ignore this term of $\frac{1}{2}$ and simply write

$$\boxed{H = \sum p^0 a^+_{\underline{p}} a_{\underline{p}}}$$

<u>Normal ordering</u> we can ensure that we discard constant offsets corresponding to vacuum expectation values by adopting the procedure called "normal ordering" denoted by ': :'

If operators are written within colons all annihilation operators are written to the right of the creation operators.

$$\therefore \; :a_p a_p^\dagger: \; = a_p^\dagger a_p .$$

$$:a_p a_p^\dagger a_p^\dagger: \; = a_p^\dagger a_p^\dagger a_p$$

Recalling that a_p annihilates the vacuum state

$$a_p|0\rangle = 0$$ which is the same as

$$\langle 0| a_p^\dagger = 0 .$$

Which means the vacuum expectation value of any normal ordered states is 0.

$$\langle 0| : \qquad : |0\rangle = 0 .$$

as at least one $a^\dagger$ or a must be adjacent to $\langle 0|$ or $|0\rangle$ respectively.

However one must not use commutation relations before the normal ordering

eg. $: a_p a_p^+ : \; = \; : \delta_{p,p'} + a_p^+ a_p : \; = \; \delta_{pp'} + a^+_{p'} p$

That shows you must simply swap a_p and a_p^+ and <u>not</u> consider the commutation relation.

$$\therefore \quad H = :\int d^3x \, \frac{1}{2} \left(\pi^2 + (\underline{\nabla} \phi)^2 + m^2 \phi^2 \right): \; = \sum p^0 N_p$$

In all Hamiltonian or Lagrangian terms "normal ordering" is assumed.

Comparing this to the P^0 term.

$$H = \frac{1}{2}\int d^3x\, \phi_i \overleftrightarrow{\partial_0} (i\partial_0\phi)$$

we can see both are of the form.

$$\frac{1}{2}\int d^3x\, \phi\, i\partial_0\, O\, \phi.$$

↳ $i\partial_0$ for H

$-i\nabla$ for $\underline{P}$.

Thus we can immediately write down.

$$\underline{P} = -:\int d^3x\, \pi \underline{\nabla} \phi: = \sum_{\underline{p}} \underline{p}\, N_p$$

by comparison with the hamiltonian term.

The Heisenberg picture immediately shows $\underline{H}$ and $\underline{P}$ are conserved.

E is H and so is immediately true and $\underline{P}$

$$-i\dot{\underline{P}} = [H, \underline{P}] = \Big[\sum_{\underline{p}} p^0 N_{\underline{p}}, \sum_{\underline{p}'} \underline{p} N_p\Big] = 0$$

since N_p commutes with all other N_p's including itself..

For the total Momentum we use the form we previously derived.

$$P^i \equiv \int d^3x\, T^{0i} = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \dot\phi} \partial^i \phi \right) = \int d^3x\, \pi \partial^i \phi$$

writing this in terms of the momentum expansion.

Firstly note the normal ordering means

$$:(\underline{\nabla} \pi)\phi: \;=\; :\phi(\underline{\nabla}\pi):$$

ie it does not matter which every way round it is written as internally it is re ordered.

Thus

$$\underline{P} = -\int d^3x\, \pi \underline{\nabla}\phi = -\frac{1}{2}\int d^3x \,(\pi \underline{\nabla}\phi + \underbrace{\pi\nabla\phi}_{\underbrace{\nabla(\pi\phi)}_{\substack{\text{surface term}\\ \to 0.}} - \underbrace{(\nabla\pi)\phi}_{\substack{\phi(\nabla\pi)\\ \text{if inside}\\ :\;:}}})$$

[Note this is the RQM flux term!!]

$$= -\frac{1}{2}\int d^3x\, (\dot\phi \underline{\nabla}\phi - \phi\nabla\dot\phi)$$

$$= -\frac{1}{2}\int d^3x\, \phi\, i \overleftrightarrow{\partial_0}(-i\nabla\phi)$$

Similarly all the components of $\underline{P}$ commute among themselves and so are all good quantum numbers.

$$[P^i, P^j] = 0$$
