Back to reality:

We assumed the Lagrangian density was a Lorentz scalar but we wrote it down for a system that had a definite rest system.

This gave an equation of motion

$$\boxed{\mu \ddot{\phi}(x) = -K \phi(x) + \tau \frac{\partial^2 \phi(x)}{\partial^2 x}.}$$

Where

$\mu = \frac{M}{\Delta x}$ (mass per unit length)

$K = \frac{k}{\Delta x}$ spring constant per unit length).

$\tau =$ stretch constant per unit length).

So how can we make this consistent with special relativity.

Set $\mu = \tau = 1$ $K = m^2$ the equation of motion of the string becomes

$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + m^2 \phi = 0 .$$

which in 3-dimensions is just the Klein-Gordon equation.

$$\left(\partial^\mu \partial_\mu + m^2\right) \phi(x) = 0 .$$

Namely, for these specific choices of parameters, the angular frequency ω and the wave number k of the plane wave solution satisfy $\omega^2 - k^2 = m^2$ and the equation of motion becomes Lorentz invariant provided the field $\phi(x)$ is scalar.

i.e. $\phi'(x') = \phi(x)$ $\quad x' = \Lambda x .$

Thus the wave exists independently of any physical thing only in the scalar field $\phi(x)$ and the invariant quantity m^2. This is exactly what is done of electromagnetic radiation.

Quantization of the Klein-Gordon field

The Lagrangian density of the K-G field is obtained by setting $\mu = \tau = 1$ $\kappa = m^2$ and extending it to 3-d.

$$\mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\left(\dot{\phi}^2 - (\nabla\phi)^2 - m^2\phi^2\right)$$

$$\equiv \quad \mathcal{L}(\phi, \partial_\mu \phi) = \frac{1}{2}\left(\partial_\mu \phi \, \partial^\mu \phi - m^2 \phi^2\right).$$

Just to check what equation of motion does this give?

$$\partial_\mu \underbrace{\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)}_{\partial^\mu \phi} - \overbrace{\frac{\partial \mathcal{L}}{\partial \phi}}^{-m^2\phi} = \left(\partial^\mu \partial_\mu + m^2\right)\phi = 0.$$

The conjugate field π to ϕ is given via the Lagrangian

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi}$$

Now for the classical Lagrangian of the springs

spring constant.

$$H = \sum \pi \dot{\phi}_i - L$$

$$= \sum \Delta x \mu \dot{\phi}^2 - \sum \Delta x \left[\frac{\mu}{2} \dot{\phi}_i^2 - \frac{k}{2} \phi_i^2 - \tau \left(\frac{\phi_i - \phi_{i-1}}{\Delta x} \right)^2 \right]$$

$$H = \int dx \underbrace{\left[\frac{\mu}{2} \dot{\phi}_i^2 + \frac{k}{2} \phi^2 + \frac{\tau}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 \right]}_{\mathcal{H}}$$

hamiltonian density.

Thus with $\mu = \tau = 1$ $k = m^2$.

$$\boxed{\mathcal{H}(\pi, \phi, \nabla\phi) = \frac{1}{2}\left(\pi^2 + (\nabla\phi)^2 + m^2\phi\right)}$$

Klein
– Gordon

Note the choice of parameters ($\mu = \eta = 1$, $k = m^2$) and the condition that $\phi(x)$ is a scalar has given us a scalar Lagrangian.

$$\mathcal{L}'(x') = \tfrac{1}{2}\Big(\underbrace{\partial'_\mu \phi'(x')\, \partial'^\mu \phi'(x')} - m^2 \phi'^2(x')\Big)$$

$$= \underbrace{\Lambda_\mu{}^\alpha \Lambda^\mu{}_\beta}_{g^\alpha{}_\beta}\, \partial_\mu \phi(x)\, \partial^\mu \phi(x)$$

$$= \tfrac{1}{2}\big(\partial_\mu \phi(x)\, \partial^\mu \phi(x) - m^2 \phi^2(x)\big) = \mathcal{L}(x).$$

<u>Now we need to quantize the Klein-Gordon field.</u>

We will turn this system into a quantum mechanical system by regarding $\phi(t,\underline{x})$ and $\pi(t,\underline{x})$ as operators in the Heisenberg picture, and introducing commutators between them.

In the discrete case:

$$\pi_i(t) = \frac{\partial L}{\partial \dot{\phi}_i} = \Delta x\, \dot{\phi}_i(t).$$

The equal time commutation relations are then well defined.

$$[\phi_i(t), \pi_i(t)] = i\delta_{ij}$$

$$[\phi_i(t), \phi_j(t)] = [\pi_i(t), \pi_j(t)] = 0.$$

The continuous limit of the conjugate field is defined directly from the Lagrangian density and given by

$$\pi(t, \underline{x}) = \dot{\phi}(t, \underline{x}).$$

$\Rightarrow$ NB. The index i is replace with $\underline{x}$, but this is not the x which defines the value of ϕ ie ϕ is a scalar quantity at that point.

Thus $\phi_i(t) = \phi(t, \underline{x}_i)$

$\pi_i(t) = \Delta x \pi(t, x_i)$

Thus $\Delta x[\phi(t, x_i), \pi(t, x_j)] = [\phi_i(t), \pi_i(t)] = \delta_{ij}$

$$\therefore \boxed{[\phi(t, x_i), \pi(t, x_j)] = \frac{\delta_{ij}}{\Delta x}}$$

We now need to consider what happens if we change from discrete variables to continuous $x_i \to x$, $x_j \to x'$.

$$\int dx\, [\phi(t,x), \pi(t,x')] \equiv \sum_i \Delta x \underbrace{[\phi(t,x_i), \pi(t,x_j)]}_{\frac{i\delta_{ij}}{\Delta x}}$$

$$= \sum i\delta_{ij} = i .$$

Thus in the continuous case the function will only pick out the value at which $x = x'$ and if that is included give a constant value. If not it gives zero.

$$\therefore \boxed{[\phi(t,x), \pi(t,x')] = i\delta(x-x')}$$

But also.

$$[\phi(t,x), \phi(t,x')] = [\pi(t,x), \pi(t,x')] = 0 .$$

Thus in general in 3 - dimensions

$$[\phi(t,\underline{x}), \pi(t,\underline{x}')] = i\,\delta^3(\underline{x}-\underline{x}') \equiv i\,\delta(x-x')\delta(y-y')\delta(z-z')$$

Note that for the Heisenberg picture at time dependence is assigned to the operator and the commutation relations are always defined at a specific time, t.

Heisenberg's Equation of Motion

We now have an infinite number of operators $\phi(t,\underline{x})$ and their conjugate operators $\pi(t,\underline{x})$.

So what equation of motion does the Heisenberg picture predict :-

$$-i\,\dot{\phi}(t,\underline{x}) = [H, \phi(\underline{x},t)]$$
$$-i\,\dot{\pi}(t,\underline{x}) = [H, \pi(t,\underline{x})]$$

simplifying by dropping the t parameter:-

$$-i\dot{\phi}(\underline{x}) = \left[\int d^3x' \left(\pi^2(\underline{x}') + (\underline{\nabla}'\phi(\underline{x}'))^2 + m^2\phi^2(\underline{x}') \right), \phi(\underline{x}) \right]$$

$$-i\dot{\pi}(\underline{x}) = \left[\int d^3x' \left(\pi^2(\underline{x}') + (\underline{\nabla}'\phi(x))^2 + m^2\phi^2(x') \right], \pi(\underline{x}) \right]$$

So how can we calculate this?

since

$$\nabla_i \phi(\underline{x}) \equiv \frac{\partial \phi(\underline{x})}{\partial x^i} \equiv \frac{\phi(\underline{x} + \epsilon\hat{\epsilon}_i) - \phi(\underline{x})}{\epsilon}$$

where $\hat{\epsilon}^i$ is the unit vector in the ith direction. Thus $\nabla\phi(x)$ can be thought of as the difference between neighbouring x, at a given t

$$[\nabla'\phi(\underline{x}'), \phi(\underline{x})] = 0$$

Since this is true for the ϕ's.

$$\int d^3x'\, f(\underline{x}')\left[\frac{\partial\phi}{\partial x^{i\prime}}(\underline{x}'),\, \pi(\underline{x})\right]$$

$$= \int d^3x'\, f(\underline{x}')\left[\frac{\phi(\underline{x}'+\epsilon\hat{e}_i)-\phi(\underline{x}')}{\epsilon},\, \pi(\underline{x})\right]$$

$$= \frac{1}{\epsilon}\int d^3x'\, f(\underline{x}')\left\{\underbrace{[\phi(\underline{x}'+\epsilon\hat{e}_i),\pi(\underline{x})]}_{i\delta^3[\underline{x}'+\epsilon\hat{\underline{e}}_i-\underline{x}]} - \underbrace{[\phi(\underline{x}'),\pi(\underline{x})]}_{i\delta^3(\underline{x}'-\underline{x})}\right\}$$

$$= \frac{i}{\epsilon}\left(f(\underline{x}-\epsilon\hat{e}_i) - f(\underline{x})\right)$$

$$= -i\,\frac{\partial f}{\partial x^i}(\underline{x}).$$

equation (a)

Thus the first Heisenberg Equation becomes:

$$-i\,\dot{\phi}(\underline{x}) = \frac{1}{2}\int d^3x'\left[\pi^2(\underline{x}') + \underbrace{(\nabla'\phi(\underline{x}'))^2}_{=0} + \underbrace{m^2\phi^2(\underline{x}'),\,\phi(\underline{x})}_{=0\;*}\right]$$

since $[\nabla'\phi(x'),\phi(x)]=0$.

$*$ since $[AB,C] = A[B,C] - [A,C]B$.

$$m^2[\phi(\underline{x}')\phi(\underline{x}'),\,\phi(x)] = \phi(\underline{x}')\underbrace{[\phi(\underline{x}'),\phi(\underline{x})]}_{=0} + \underbrace{[\phi(x')\phi(x)]}_{=0}\phi(x')$$

$$= \frac{1}{2}\int d^3x' \left[\pi^2(\underline{x}'), \phi(\underline{x})\right]$$

so using the same $\underbrace{\text{commutator}}$ expansion:-

$$\pi(\underline{x}')\underbrace{\left[\pi(\underline{x}'), \phi(\underline{x})\right]}_{-i\delta^3(\underline{x}-\underline{x}')} + \underbrace{\left[\pi(\underline{x}'), \phi(\underline{x})\right]}_{-i\delta^3(\underline{x}-\underline{x}')}\pi(\underline{x}')$$

$$= -i\,\pi(\underline{x})$$

$\therefore$ we get

$$\boxed{\dot{\phi}(t,\underline{x}) = \pi(t,\underline{x})}$$

The second equation becomes.

$$-i\dot{\pi}(\underline{x}) = \frac{1}{2}\int d^3x' [\underbrace{\pi^2(\underline{x}')}_{0} + (\nabla'\phi(\underline{x}'))^2 + m^2\phi^2(\underline{x}'), \pi(x)]$$

since $[\pi(\underline{x}'), \pi(x)] = 0$.

$$= \underbrace{\frac{1}{2}\int d^3x' [(\nabla'\phi(\underline{x}'))^2, \pi(\underline{x})]}_{} + \underbrace{\frac{m^2}{2}\int d^3x' [\phi^2(\underline{x}'), \pi(\underline{x})]}_{}$$

$$\frac{1}{2}\sum_i \int d^3x' [\underbrace{\partial_i'\phi(\underline{x}')[\partial_i'\phi(\underline{x}'), \pi(\underline{x})]}_{-i\partial_i\partial_i\phi(\underline{x})}$$

$$+ \underbrace{[\partial_i\phi(\underline{x}'), \pi(\underline{x})]\partial_i'\phi(\underline{x}')}_{-i\partial_i\partial_i\phi(\underline{x})}$$

$$= \frac{1}{2}\sum_i 2 - i\partial_i\partial_i\phi(\underline{x})$$

$$= -i\nabla^2\phi(x)$$

$$\frac{m^2}{2}\int d^3x' \{\phi(\underline{x}')\underbrace{[\phi(\underline{x}'), \pi(\underline{x})]}_{i\delta(\underline{x}'-\underline{x})} + \underbrace{[\phi(\underline{x}'), \pi(x')]}_{i\delta(\underline{x}'-\underline{x})}\phi(x')$$

$$= i m^2\phi(\underline{x}).$$

Thus this gives

$$\dot{\pi}(t,\underline{x}) = \nabla^2 \phi(t,\underline{x}) - m^2 \phi(t,x).$$

$$\pi(t,x) = \dot{\phi}(t,x).$$

Putting them together

$$\ddot{\phi}(t,\underline{x}) = \nabla^2 \phi(t,\underline{x}) - m^2 \phi(t,\underline{x})$$

$$\boxed{\Rightarrow \left(\partial^2 + m^2\right) \phi(t,\underline{x}) = 0 .}$$

Thus we have shown that the Klein-Gordon equation comes from the Heisenberg equations of motion for $\phi(t,x)$, $\pi(t,x)$ and the commutation relations. Thus $\phi(t,x)$ is shown to have satisfied K.G. equation.

N.B. This is not so much a statement about K.G. as the $\mathcal{H}$ was defined for $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \, \partial^\mu \phi - m^2 \phi^2)$ and the Euler-Lagrange equations give us K.G. It is more a statement that the $\phi(t,x)$ with the commutation relations satisfy the K.G. equation.

The momentum expansion :-

We have derived the operator field $\phi(t, \underline{x})$. We will now express these in terms of normal modes, which will give rise to creation and annihilation operators.

If we place our fields in a large box

$$-L/2 < x_i < L/2 \qquad (i = 1, 2, 3).$$

which imposes a periodicity condition on the field

$$\phi(t, x+L, y, z) = \phi(t, x, y, z) \qquad \text{etc.}$$

At any time t any operator field can be expanded as

$$\phi(t, \underline{x}) = \sum_{p} C_p(t)\, e^{i \underline{p} \cdot \underline{x}}$$

where $C_p(t)$ are operators and p take discrete values that satisfy

$$\underline{p} = (p_x, p_y, p_z) = \left(\frac{2\pi n_x}{L}, \frac{2\pi n_y}{L}, \frac{2\pi n_z}{L} \right)$$

$$-\infty < n_x, n_y, n_z < \infty .$$

The set of functions $e^{i\underline{p}\cdot\underline{x}}$ forms a complete and orthonormal set.

ie

1) $$\int d^3x \left(e^{i\underline{p}\cdot\underline{x}}\right)^*\left(e^{+i\underline{p}'\cdot\underline{x}}\right) = V\,\delta_{\underline{p},\underline{p}'}$$

or $V\,\delta(\underline{p}-\underline{p}')$ if we consider p continuous.

2) $$\sum_{\underline{p}} \left(e^{i\underline{p}\cdot\underline{x}}\right)^* e^{i\underline{p}\cdot\underline{x}'} = V\,\delta(x-x').$$

for 1) note that if $p = p'$ then $\equiv \int d^3x = V$.

if not $\underline{p}-\underline{p}'$ is finite and the integral oscillates; (see proof on page 26).

2) Convert $\sum_{p'} = \frac{V}{(2\pi)^3}\int d^3p \quad (L\to\infty)$.

Each mode takes a volume $\left(\frac{2\pi}{L}\right)^3$

$$\therefore \int d^3p\, f(\underline{p}) = \sum_p \frac{dv\, f(p)}{(2\pi/L)^3} = \left(\frac{2\pi}{L}\right)^3 \sum_p f(\underline{p})$$

Thus we have

$$\sum_{p'} \left(e^{i\underline{p}\cdot\underline{x}}\right)^* e^{i\underline{p}\cdot\underline{x}} = \frac{V}{(2\pi)^3}\underbrace{\int d^3p\, e^{i\underline{p}\cdot(\underline{x}-\underline{x}')}}_{(2\pi)^3\,\delta^3(\underline{x}-\underline{x}')}$$

$$= V\delta^3(\underline{x}-\underline{x}')$$

NB.

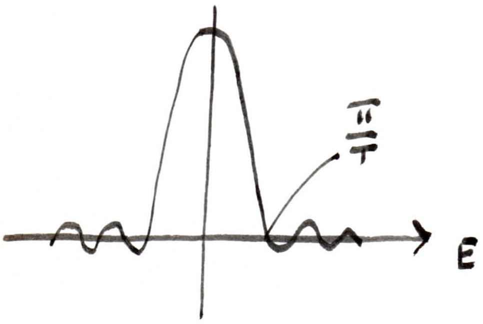
Remember

$$\int_{-\infty}^{\infty} e^{-iEt}\, dt = \lim_{T\to\infty} \int_{-T}^{T} e^{-iEt}\, dt$$

$$= \lim_{T\to\infty} \left[\frac{e^{-iEt}}{-iE} \right]_{-T}^{T}$$

$$= \lim_{T\to 0} \left[i\, \frac{e^{-iET} - e^{iET}}{E} \right]$$

$$= \lim_{T\to\infty} 2i\, \frac{\sin ET}{E}$$



$$\lim_{T\to\infty} \int_{-\infty}^{\infty} \frac{\sin ET}{E}\, dE = \lim_{T\to\infty} \mathrm{Im} \int_{-\infty}^{\infty} \frac{e^{iz}}{z}\, dz$$

$z = ET$.

theorem of residues.

$$\oint_c \frac{f(z)}{z - z_0}\, dz = \pi i f(z_0)$$

Thus $\int_{-\infty}^{\infty} \frac{\sin ET}{E}\, dE = \pi$

Thus at $\lim T\to\infty$ 0 everywhere except $T = 0$

and.

$$\int_{-\infty}^{\infty} \frac{\sin ET}{E}\, dE = \pi \quad \text{a constant} \equiv \delta(E).$$

So given $\sum_{\underline{p}} (e^{i\underline{p}\cdot\underline{x}})^* e^{i\underline{p}\cdot\underline{x}'}$

$$= \frac{V}{(2\pi)^3} \underbrace{\int d^3p \, e^{ip.(x'-x)}}_{(2\pi)^3 \delta^3(\underline{x}-x')}$$

$$= V \delta^3(\underline{x}-x') . \quad \text{(ie orthogonal in } x.)$$

Now $\phi(t,\underline{x}) = \sum_{\underline{p}} c_{\underline{p}}(t) \, e^{i\underline{p}.\underline{x}}$.

This is a <u>unique</u> expansion since

$$\int d^3x \, (e^{i\underline{p}.\underline{x}})^* \phi(t,x) = \sum_{\underline{p}} c_{\underline{p}}(t) \underbrace{\int d^3x \, (e^{i\underline{p}.\underline{x}})^* e^{i(\underline{p}'.\underline{x})}}_{V\delta_{pp'}}$$

$$= V c_{\underline{p}}(t) .$$

$$\therefore \quad \boxed{c_p = \frac{1}{V}\int d^3x \, (e^{-ip.x})^* \phi(t,\underline{x})}$$

Now we can show it a complete set of functions.

$$\sum_{\underline{k}} C_{\underline{k}} e^{i\underline{k}\cdot\underline{x}} = \frac{1}{V}\sum_{\underline{k}} \int d^3x' \left(e^{i\underline{k}\cdot\underline{x}'}\right)^* \phi(t,\underline{x}') e^{i\underline{k}\cdot\underline{x}}$$

$$= \frac{1}{V}\int d^3x' \underbrace{\sum_{\underline{k}} \left(e^{i\underline{k}\cdot\underline{x}'}\right)^* \left(e^{i\underline{k}\cdot x}\right)}_{V\delta^3(x-x')} \phi(t,x')$$

$$= \phi(t,x).$$

Thus it must be complete.

Since ϕ is an operation field and if it represents an observable:

$$\phi^{+} = \phi \longrightarrow \underbrace{\sum_{p} c_{p}^{\dagger} e^{-ip.x}}_{\sum_{p} c_{-p}^{\dagger}(t)\, e^{ip.x}} = \sum_{p} c_{p}(t) e^{ip.x}$$

$(p \to -p)$.

Thus the hermiticity condition is

$$\boxed{c_{-p}^{+}(t) = c_{p}(t)}$$

If we consider the field expanded as before

$$\phi(t, x') = \sum_{\underline{p}'} c_{\underline{p}}(t)\, e^{i\underline{p}\cdot x}$$

and substituting this into

$$\ddot{\phi} = \nabla^2 \phi - m^2 \phi$$

$$\therefore \sum_{\underline{p}} \ddot{c}_{\underline{p}}(t)\, e^{i\underline{p}\cdot\underline{x}} = \sum_{\underline{p}} \left[(i\underline{p})^2 - m^2 \right] c_{\underline{p}}(t)\, e^{i\underline{p}\cdot\underline{x}}$$

$$\Rightarrow \boxed{\ddot{c}_{\underline{p}}(t) = -p^{0\,2}\, c_{\underline{p}}}$$

where $p^0 \equiv \sqrt{\underline{p}^2 + m^2} \geq 0$ $\boxed{\text{Definition}}$

The general solution is therefore.

$$c_p(t) = A_{\underline{p}}\, e^{-ip^0 t} + B_{\underline{p}}\, e^{ip^0 t}.$$

where $A_{\underline{p}}$ and $B_{\underline{p}}$ are constant operators (ie independent of t).

So what of the condition of hermaticity

$$C^{+}_{-p}(t) = C_p(t).$$

$$A^{+}_{-p}\, e^{ip^0 t} + B^{+}_{-p}\, e^{-ip^0 t} = A_p\, e^{-ip^0 t} + B_p\, e^{ip^0 t}$$

Thus $\boxed{A^{+}_{-p} = B_p}$

$\boxed{B^{+}_{-p} = A_p}$

However they are the same conditions

So we can find the general solution for the Klein - Gordon equation by substituting

$$C_p(t) = A_p\, e^{-ip_0 t} + B_p\, e^{+ip^0 t}$$

using the condition $A^{+}_{-p} = B_p$

$$\phi(t,\underline{x}) = \sum_{\underline{p}} \left(A_{\underline{p}} e^{-ip^0 t} + A^+_{-\underline{p}} e^{ip^0 t} \right) e^{i\underline{p}\cdot\underline{x}}$$

$$= \sum_{\underline{p}} \Big(A_{\underline{p}} e^{-ip^0 t + i\underline{p}\cdot\underline{x}}$$

$$+ \underbrace{A^+_{-\underline{p}} e^{ip^0 t + i\underline{p}\cdot\underline{x})}}_{\substack{\text{relabel } p \to -\underline{p} \\ \text{(its just a label)}.}} \Big)$$

$$\boxed{\phi(t,x) = \sum_{\underline{p}} \left(A_{\underline{p}} e^{-ip^\mu x_\mu} + A^+_{\underline{p}} e^{ip.x} \right).}$$

Now define the operator $a_{\underline{p}}$

$$\boxed{a_{\underline{p}} \ni \qquad A_{\underline{p}} = \frac{1}{\sqrt{2p^0 V}} \, a_{\underline{p}}}$$

The normalization factor $1/\sqrt{2p^0 V}$ is defined so that the $[\phi, \pi]$ gives.

$$[a_{\underline{p}}, a^+_{\underline{p}'}] = \delta_{pp'}$$