Noether Currents PART II

For each transformation that leaves the action invariant there exists a conserved quantity.

Proof:

The value of the Lagrangian density $\mathcal{L}$ is uniquely defined at each space point x^μ. $\mathcal{L}(\phi, \partial_\mu \phi)$ $\phi(x^\mu)$

The difference between the values of $\mathcal{L}$ at x^μ and at $x^\mu + \epsilon^\mu$ is given by:

$$\delta \mathcal{L} = \left(\partial_\mu \mathcal{L}\right) \epsilon^\mu . \qquad \left(\text{ie } \epsilon^\mu \equiv \delta x^\mu\right)$$

$$\delta\phi \equiv \phi(x+\epsilon) - \phi(x) = (\partial_\nu \phi)\,\epsilon^\nu$$

$$\delta(\partial_\mu\phi) \equiv \partial_\mu \phi(x+\epsilon) - \partial_\mu \phi(x) = \partial_\nu(\partial_\mu\phi)\,\epsilon^\nu$$

$$\delta \mathcal{L} = \underbrace{\frac{\partial \mathcal{L}}{\partial \phi}}_{\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right)} \overbrace{\delta\phi}^{(\partial_\nu\phi)\,\epsilon^\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \overbrace{\delta(\partial_\mu\phi)}^{\partial_\nu(\partial_\mu\phi)\,\epsilon^\nu}$$

This comes from the Euler - Lagrange equations.

$$= \left[\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right)\partial_\nu\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\mu\partial_\nu\phi\right]\epsilon^\nu$$

$$= \partial_\mu\left[\frac{\partial \mathcal{L}}{\partial(\partial^\mu\phi)}\,\partial_\nu\phi\right]\epsilon^\nu$$

note we have used Euler - Lagrange. and so $\phi(x)$ must be physical

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi \right) \epsilon^\nu = \partial_\mu \mathcal{L}\, \epsilon^\mu$$
$$= \partial_\mu \mathcal{L}\, g^\mu{}_\nu \epsilon^\nu$$

Thus $$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} g^\mu{}_\nu \right) \epsilon^\nu = 0.$$

since ϵ^ν is arbitrary we must have

$$\partial_\mu J^{\mu\nu} = 0$$

where
$$\boxed{J^{\mu\nu} = \left(\frac{\delta \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi - \mathcal{L} g^\mu{}_\nu \right)}$$

These are called the Noether Currents of space-time translations.

The space integral of the time component of each current ($\nu = 1, 4$) is conserved.

$$\partial_0 J^{0\nu} = -\partial_i J^{i\nu}$$

$$\frac{\partial}{\partial t}\left(\int d^3x\, J^{0\nu} \right) = \int d^3x\, \partial_0 J^{0\nu} = -\int d^3x\, \partial_i J^{i\nu}$$
$$= -\int_A da_i\, J^{i\nu} = 0.$$

The last integral is over an infinite surface boundary where $J^{i\nu} \to 0$.

Thus the four quantities

$$P^\nu \equiv \int d^3x \, T^{0\nu} \quad (\nu = 0, 1, 2, 3)$$

are conserved.

P^0 is just the hamiltonian

$$P^0 \equiv \int d^3x \, T^{00} = \int d^3x \underbrace{\left(\overbrace{\frac{\partial \mathcal{L}}{\partial \dot\phi}}^{\pi} \dot\phi - \mathcal{L} \right)}_{\mathcal{H}} = H$$

It is natural to identify

$$P^i \equiv \int d^3x \, T^{0i} = \int d^3x \left(\frac{\partial \mathcal{L}}{\partial \dot\phi} \partial^i \phi \right)$$

$$= \int d^3x \; \pi \, \partial^i \phi$$

as the three momentum components.

If we show that P^ν is a 4-vector then P^i must be the 3-momentum.

[Ignore this section on a first read].

To show P^ν is a 4-vector.

Given that $\mathcal{L}(x)$ is a scalar.

$$\mathcal{L}'(x') = \mathcal{L}(x) \qquad x' = \Lambda x.$$

note $\delta\mathcal{L} = \partial_\mu J^{\mu\nu} \epsilon^\nu$

Since $\partial_\mu, \epsilon^\nu$ are 4-vectors if $\delta\mathcal{L}$ is to be a scalar $J^{\mu\nu}$ must transform as a tensor (a dyadic)

$$\therefore J'^{\mu\nu}(x') = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta J^{\alpha\beta}(x)$$

Since P'^ν and P^ν are constants we can chose to evaluate them at $t=0$. and $t'=0$.

$$P'^\nu = \int d^3x' \, J'^{0\nu}(0, \underline{x}).$$

$$P^\nu = \int d^3x \, J^{0\nu}(0, \underline{x})$$

Now for a proper transformation

$$d^4x' = \underbrace{\det\Lambda}_{+1.} d^4x \qquad \therefore \qquad d^4x' = d^4x.$$

Now a property of δ- function

$$\delta(f(s)) = \sum_i \frac{1}{|f'(s)|} \delta(s - s_i)$$

s_i roots of $f(s) = 0$.

proof.

$$\int \delta(x)\, f(x)\, dx = f(0)$$

$$\Rightarrow \int \delta(f(s))\, g(f(s))\, df(s) = \sum_i g(f(s) = 0)$$

$$\Rightarrow \int \delta(f(s))\, g(f(s)) \frac{df(s)}{ds} ds = \sum g(f(s) = 0)$$

$$= \sum \int g\, \delta(s - s_i)\, ds$$

Thus by inspection

$$\delta(f(s)) \frac{df(s)}{ds} = \sum_i \delta(s - s_i)$$

$$\boxed{\therefore \delta(f(s)) = \frac{\sum_i \delta(s - s_i)}{|f'(s_i)|}}$$

Since $t' = \Lambda^{00} t + \Lambda^0{}_i x^i \qquad \dfrac{dt'}{dt} = \Lambda^0{}_0$

$$\therefore \quad \delta(t') = \delta(\Lambda^0{}_\rho x^\rho)$$

$$= \frac{1}{\Lambda^0{}_0}\,\delta\left(t + \frac{\Lambda^0{}_i x^i}{\Lambda^0{}_0}\right)$$

$$\text{Now } P'^{\nu} \equiv \int \underbrace{d^3x'}_{d^3x} \, J'^{0\nu}(0, \underline{x}') \equiv \int d^3x \, \overbrace{\delta(t')}^{\frac{1}{\Lambda^0{}_0}\delta\left(t + \frac{\Lambda^0{}_i x^i}{\Lambda^0{}_0}\right)} \underbrace{J'^{0\nu}(t', \underline{x}')}_{\Lambda^0{}_\alpha \Lambda^\nu{}_\beta J^{\alpha\beta}(t, \underline{x})}$$

$$\equiv \frac{1}{\Lambda^0{}_0} \int dt \, d^3x \, \delta\left(t + \frac{\Lambda^0{}_i x^i}{\Lambda^0{}_0}\right) \Lambda^0{}_\alpha \Lambda^\nu{}_\beta J^{\alpha\beta}(t, \underline{x})$$

$$= \frac{1}{\Lambda^0{}_0} \int d^3x \, \Lambda^0{}_\alpha \Lambda^\nu{}_\beta J^{\alpha\beta}\left(-\frac{\Lambda^0{}_i x^i}{\Lambda^0{}_0}, \underline{x}\right)$$

We can write $\Lambda^\mu{}_\nu$ as an infinitesimal transformation

no change

$$\Lambda^\mu{}_\nu = g^\mu{}_\nu + \omega^\mu{}_\nu$$

not diagonal.

small change

leading diagonal

if $\Lambda^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu}$.

now the lorentz transfomation transforms as.

$$\Lambda_{\nu\alpha} \Lambda^{\nu}{}_{\beta} = g_{\alpha\beta}.$$

$$\therefore \quad g_{\alpha\beta} = \Lambda_{\nu\alpha} \Lambda^{\nu}{}_{\beta}$$

$$= (g_{\nu\alpha} + \omega_{\nu\alpha}) \, (g^{\nu}{}_{\beta} + \omega^{\nu}{}_{\beta})$$

$$= g_{\nu\alpha} g^{\nu}{}_{\beta} + \omega_{\nu\alpha} g^{\nu}{}_{\beta} + g_{\nu\alpha} \omega^{\nu}{}_{\beta} + \omega_{\nu\alpha} \omega^{\nu}{}_{\beta}.$$

$$= g_{\alpha\beta} + \omega_{\beta\alpha} + \omega_{\alpha\beta} + \omega_{\nu\alpha} \omega^{\nu}{}_{\beta}.$$

$$\therefore \quad \boxed{\omega_{\beta\alpha} = -\omega_{\alpha\beta}.}$$

$$\therefore \quad \omega_{\alpha\beta} = \begin{pmatrix} 0 & \omega_{01} & \omega_{02} & \omega_{03} \\ -\omega_{01} & 0 & \omega_{12} & \omega_{13} \\ -\omega_{02} & -\omega_{12} & 0 & \omega_{23} \\ -\omega_{03} & -\omega_{13} & -\omega_{23} & 0 \end{pmatrix}$$

$$\boxed{\Lambda^{\mu}{}_{\nu} = g^{\mu}{}_{\nu} + \omega^{\alpha}{}_{\beta}.} \qquad \omega^{\alpha}{}_{\beta} = g^{\alpha i} \omega_{i\beta}$$

$$= \begin{pmatrix} 0 & \omega_{01} & \omega_{02} & \omega_{03} \\ \omega_{01} & 0 & -\omega_{12} & -\omega_{13} \\ \omega_{02} & \omega_{12} & 0 & -\omega_{23} \\ \omega_{03} & \omega_{13} & \omega_{23} & 0 \end{pmatrix}.$$

$\therefore$ we have certain symmetries

$$\omega^{\alpha}{}_{\beta} = \omega^{\beta}{}_{\alpha} \quad \text{if} \quad \alpha = 0 \quad \beta = 1 \quad \text{etc.}$$

$$\omega^{\alpha}{}_{\beta} = -\omega^{\beta}{}_{\alpha} \quad \text{if} \quad \alpha = 1 \quad \beta = 2 \quad \text{etc.}$$

$$\omega^{0}{}_{i} J \qquad - \partial_j J$$

$$= \underbrace{\int d^3x \, J^{0\nu}} + \omega^{\nu}{}_{\beta} \underbrace{\int d^3x \, J^{0\beta}} + \int d^3x \, \omega^{0}{}_{i} J^{i\nu} + \int d^3x \, \omega^{0}{}_{i} \underbrace{(\partial_j J^{j\nu})}_{\partial_j (J^{j\nu} x^i) - J^{j\nu} \partial_j x^i} x^i$$