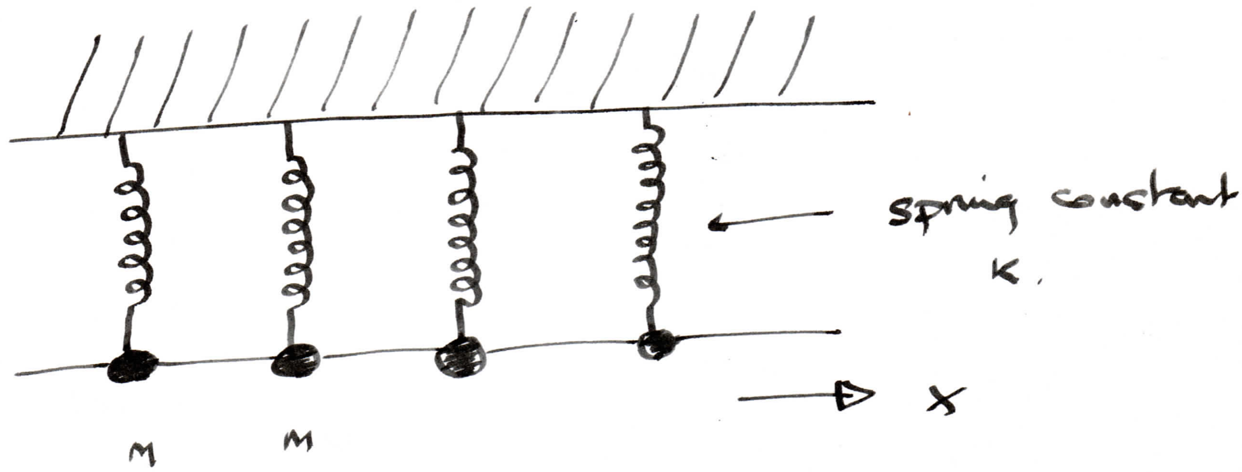
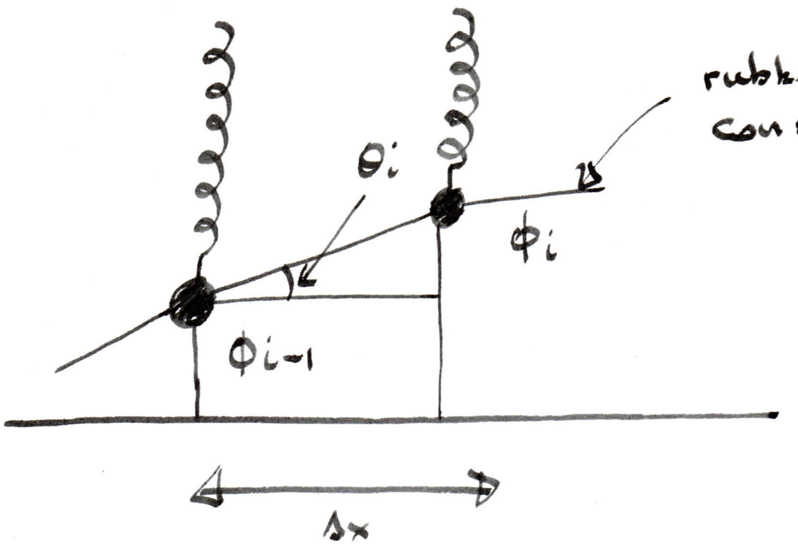
Lagrangian Formulation of a Classical Wave

The first step towards quantizing a field is to find the Lagrangian of a field system, then the canonical momenta can be found, and the Hamiltonian formed. Then we introduce commutation relations between the coordinates and momenta which quantizes the system.

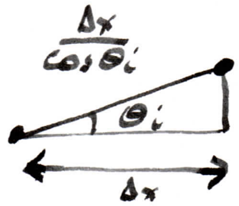


Consider a string of masses (each mass m).



rubber with stretch constant τ.

ϕ_i are the displacements.



for small θ_i's:-

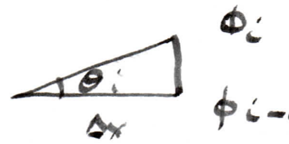
$$V_{rubber} = \sum_i \tau \Delta x \left(\frac{1}{\cos\theta_i} - 1 \right)$$

$$= \sum_i \tau \Delta x \left(\frac{1 - \cos\theta_i}{\cos\theta_i} \right) \qquad \cos\theta_i \simeq 1.$$

$$\simeq \sum \tau \Delta x \left(1 - \cos\theta_i \right)$$

$$= \sum \tau \Delta x \, \frac{1}{2} \sin^2\theta_i$$

$$\simeq \sum \tau \Delta x \frac{\theta_i^2}{2}$$



$\tan\theta_i \simeq \theta_i = \frac{\phi_i - \phi_{i-1}}{\Delta x}.$

$$= \sum \tau \Delta x \, \frac{1}{2} \left(\frac{\phi_i - \phi_{i-1}}{\Delta x} \right)^2$$

$$V_{springs} = \sum_i \frac{1}{2} k \phi_i^2$$

The Lagrangian is then.

$$L(\underline{\phi}, \underline{\dot{\phi}}) = T - V_{spring} - V_{rubber}$$

$$= \sum \frac{1}{2} m \dot{\phi}_i^2 - \frac{1}{2} k \phi_i^2 - \frac{1}{2} \tau \Delta x \left(\frac{\phi_i - \phi_{i-1}}{\Delta x} \right)^2$$

We can change explicit mass and spring constant to mass μ and spring constant K per unit length.

$$m = \mu \Delta x \qquad k = K \Delta x$$

$$L(\underline{\phi}, \underline{\dot{\phi}}) = \sum_i \Delta x \left[\frac{\mu}{2} \dot{\phi}_i^2 - \frac{K}{2} \phi_i^2 - \frac{\tau}{2} \left(\frac{\phi_i - \phi_{i-1}}{\Delta x} \right)^2 \right]$$

The Lagrange equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_i}\right) = \frac{\partial L}{\partial \phi_i} \qquad (i = 1 \ldots n)$$

$$\therefore \quad \mu \ddot{\phi}_i = -K\phi_i - \tau \, \frac{(\phi_i - \phi_{i-1}) - (\phi_{i+1} - \phi_i)}{\Delta x^2}$$

Where the tension term in $\partial L / \partial \phi_i$ resulted from two terms corresponding to i and $i+1$.
This is a set of n differential equations that are coupled.

Taking the limit $n \to \infty$ and $\Delta x \to 0$ while keeping μ and K constant. In this limit we have

$$\frac{(\phi_i - \phi_{i-1})}{\Delta x} = \frac{\partial \phi}{\partial x}(x_i)$$

the tension term can be written

$$\frac{(\phi_i - \phi_{i-1}) - (\phi_{i+1} - \phi_i)}{(\Delta x)^2} = \frac{\frac{(\phi_i - \phi_{i-1})}{\Delta x} - \frac{(\phi_{i+1} - \phi_i)}{\Delta x}}{\Delta x}$$

$$= \frac{\frac{\partial \phi}{\partial x}(x_i) - \frac{\partial \phi}{\partial x}(x_{i+1})}{\Delta x} = -\frac{\partial^2 \phi}{\partial x^2}(x_i).$$

So the equations of motion are written

$$\mu \ddot{\phi}(x) = -\kappa \phi(x) + \tau \frac{\partial^2 \phi}{\partial x^2}(x).$$

In the limit $n \to \infty$ $\Delta x \to 0$. the Lagrangian can be written as an integral over x.

$$\boxed{\begin{aligned} L(\underline{\phi}, \underline{\dot{\phi}}) &= \int dx\, \mathcal{L}\left(\phi, \dot{\phi}, \frac{\partial \phi}{\partial x}\right) \\ \text{where} \quad \mathcal{L}\left(\phi, \dot{\phi}, \frac{\partial \phi}{\partial x}\right) &= \frac{\mu}{2}\dot{\phi}^2 - \frac{\kappa}{2}\phi^2 - \frac{\tau}{2}\left(\frac{\partial \phi}{\partial x}\right)^2. \end{aligned}}$$

The range of integration for the action is either $\pm\infty$ or $\pm L/2$, and in each case $\phi(\pm\infty) = 0$, or $\phi(x+L) = \phi(x)$, so that boundary conditions do not contribute.

$$\boxed{\mathcal{L}(\phi, \dot{\phi}, \partial\phi/\partial x) \text{ is called the Lagrangian Density}}$$

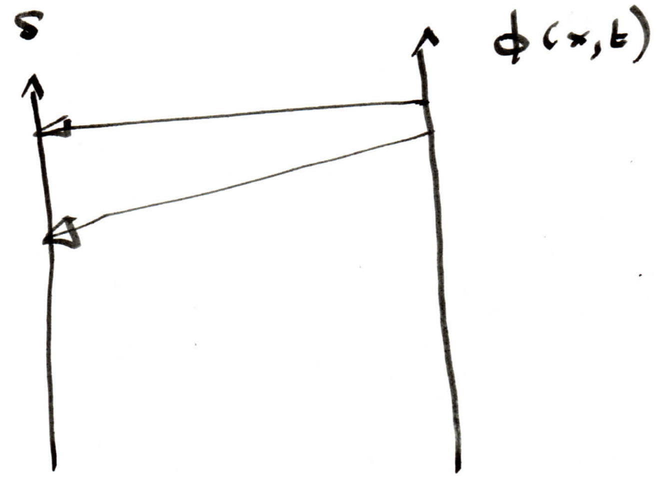
Up to now the equation of motion was found by taking $\Delta x \to 0$ for the equations of the discrete particles.

Lets now find how to get these equations of motion directly from the Lagrangian Density

$$S = \int_{t_1}^{t_2} L\,dt = \int_{t_1}^{t_2} dt \int dx \; \mathcal{L}(\phi, \dot{\phi}, \frac{\partial\phi}{\partial x})$$

The function $\phi(x,t)$ specifies a motion of the string (physical, or unphysical) with which there is an associate number S.

The action (real number) S is thus a mapping of a function $\phi(x,t)$ onto a number S. This is called a functional.



The action principle tells us that the true motion is the one with the smallest S.

if we vary the function slightly around the true motion

$$\phi'(t,x) = \phi(t,x) + \delta\phi(t,x).$$

with the condition $\delta\phi(t,x) = \delta\phi(t,x) = 0$.

The condition for minimums should be

$\delta S = S' - S = 0$ to first order in $\delta\phi$

(ie $\delta S / \delta\phi = 0$)

$$\delta S = \int dt\, dx\, (\mathcal{L}' - \mathcal{L}) = \int dt\, dx\, \delta\mathcal{L} = 0$$

$$\delta \mathcal{L}(t,x) = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial \dot\phi}\delta\dot\phi + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)}\delta\left(\frac{\partial\phi}{\partial x}\right)$$

$$\delta\phi = \phi' - \phi \qquad \delta(\dot\phi) = \dot\phi' - \dot\phi \qquad \delta\left(\frac{\partial\phi}{\partial x}\right) = \frac{\partial\phi'}{\partial x} - \frac{\partial\phi}{\partial x}$$

$$= \frac{\partial\,\delta\phi}{\partial x}.$$

Then using $A\frac{\partial}{\partial s}B = \frac{\partial}{\partial s}(AB) - \left(\frac{\partial}{\partial s}A\right)B.$

$$\frac{\partial\mathcal{L}}{\partial\dot\phi}\underbrace{\delta(\dot\phi)}_{\delta\dot\phi} = \underbrace{\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot\phi}\delta\phi\right)}_{0} - \left(\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial\dot\phi}\right)\right)\delta\phi$$

$$\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)}\underbrace{\delta\left(\frac{\delta\phi}{\partial x}\right)}_{\frac{\partial}{\partial x}\delta\phi} = \underbrace{\frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)}\delta\phi\right)}_{0} - \left(\frac{\partial}{\partial x}\left(\frac{\partial\mathcal{L}}{\partial\left(\frac{\partial\phi}{\partial x}\right)}\right)\right)\delta\phi$$

where the left hand terms on the right hand side are zero because $\delta\phi(t_1, x) = \delta\phi(t_2, x) = 0$ when integrated

Thus

$$\delta S = \int dt\, dx \left[\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) - \frac{\partial}{\partial x} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial x}\right)} \right] \delta\phi \;\cancel{+}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\delta\phi\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial x}\right)}\delta\phi\right)$$

$$\int \left[\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\delta\phi\right]_{t_1}^{t_2} dx \qquad\qquad \int_{t_1}^{t_2} \left[\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial x}\right)}\delta\phi\right]_{-\infty}^{\infty} dt$$

So

$$\delta S = \int dt\, \delta x \left[\frac{\partial \mathcal{L}}{\partial \phi} - \frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial x}\right)} \right]\delta\phi$$

since δS must be zero for all (t,x) then

$$\boxed{\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial x}\right)}\right) = \frac{\partial \mathcal{L}}{\partial \phi}}$$

This is called the Euler - Lagrange equation for the Lagrangian density.

Applying this to the Lagrangian density

$$\mathcal{L}\left(\phi, \dot{\phi}, \frac{\partial \phi}{\partial x}\right) = \frac{\mu}{2}\dot{\phi}^2 - \frac{k}{2}\phi^2 - \frac{\tau}{2}\left(\frac{\partial \phi}{\partial x}\right)^2$$

$$\Rightarrow \quad \frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) = \mu\ddot{\phi} \quad ; \quad \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial x}\right)}\right) = -\tau \frac{\partial^2 \phi}{\partial x^2}$$

$$\boxed{\mu\ddot{\phi} - \tau \frac{\partial^2 \phi}{\partial x^2} = -k\phi}$$

3 - dimensions

$$\mathcal{L}(\phi, \dot{\phi}, \underline{\nabla}\phi) = \mathcal{L}(\phi, \partial_\mu \phi)$$

So adding terms in

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial \dot{\phi}}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial x}\right)}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial y}\right)}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{\partial \phi}{\partial y}\right)}\right) = \frac{\partial \mathcal{L}}{\partial \phi}.$$

$$\equiv \boxed{\partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = \frac{\partial \mathcal{L}}{\partial \phi}.}$$

More than one field

$$\mathcal{L}(\tilde{\phi}\ \partial_\mu \tilde{\phi})\ ; \quad \tilde{\phi}(x) \equiv (\phi_1(x), \ldots, \phi_N(x)), \ x^\mu = (t, x)$$

using the same conditions.

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_k)} \right) = \frac{\partial \mathcal{L}}{\partial \phi_k} \qquad (k = 1, \ldots N).$$

Hamiltonian Density

In the system where we only have one field $\phi(x, t)$ $(k = 1)$.

Starting from the one-dimensional discrete system whose total Lagrangian was given by $L(\phi, \dot{\phi})$ with $\phi = \phi_i\ (i = 1 \ \ n)$. Note i labels position along the string, k different fields ϕ_k at each point.

$$\boxed{\pi_i \underset{\text{def}}{\equiv} \frac{\partial L}{\partial \dot{\phi}_i} = \Delta x \mu \dot{\phi}_i}$$

So by definition the Hamiltonian is

$$H \equiv \sum_i \pi_i \dot{\phi}_i - L$$

$$= \sum \Delta x \mu \dot{\phi}_i^2 - \sum \Delta x \left[\frac{\mu}{2} \dot{\phi}_i^2 - \frac{\kappa}{2} \phi_i^2 - \frac{\tau}{2} \left(\frac{\phi_i - \phi_{i-1}}{\Delta x} \right)^2 \right]$$

$$= \sum_i \Delta x \left[\frac{\mu}{2} \dot{\phi}_i^2 + \frac{\kappa}{2} \phi_i^2 + \frac{\tau}{2} \left(\frac{\phi_i - \phi_{i-1}}{\Delta x} \right)^2 \right]$$

In the continuous limit.

$$\int dx \left[\frac{\mu}{2} \dot{\phi}^2 + \frac{\kappa}{2} \phi^2 + \frac{\tau}{2} \left(\frac{\partial \phi}{\partial x} \right)^2 \right]$$

But now using the Lagrangian density.

$\mathcal{L}(\phi, \dot{\phi}, \partial\phi/\partial x)$

$$\boxed{\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu \dot{\phi}(x)}$$

and

$$\mathcal{H}\left(\pi, \phi, \frac{\partial \phi}{\partial x}\right) \equiv \pi \dot{\phi} - \mathcal{L}\left(\phi, \dot{\phi}, \frac{\partial \phi}{\partial x}\right)$$

$$= \frac{\mu}{2} \dot{\phi}^2 + \frac{k}{2} \phi^2 + \frac{\tau}{2} \left(\frac{\partial \phi}{\partial x}\right)^2 .$$

With multiple fields

$$\tilde{\pi}_k(x) \equiv \frac{\partial \mathcal{L}(\tilde{\phi}, \partial_\mu \tilde{\phi})}{\partial \dot{\phi}_k} \qquad k = (1, \dots N)$$

$$\boxed{H(\tilde{\pi}, \tilde{\phi}, \nabla \tilde{\phi}) = \sum_k \pi_k \dot{\phi}_k - \mathcal{L} .}$$