Quantum Field Theory PART 1

Relativistic Quantum Mechanics described the state of a single particle either Klein-Gordon or Dirac. The wave function has only a single pair of conjugate variables.

In Quantum Field Theory we associate such a pair of conjugate variables with every such point in a field. This allows for the field to described an infinite number of particles.

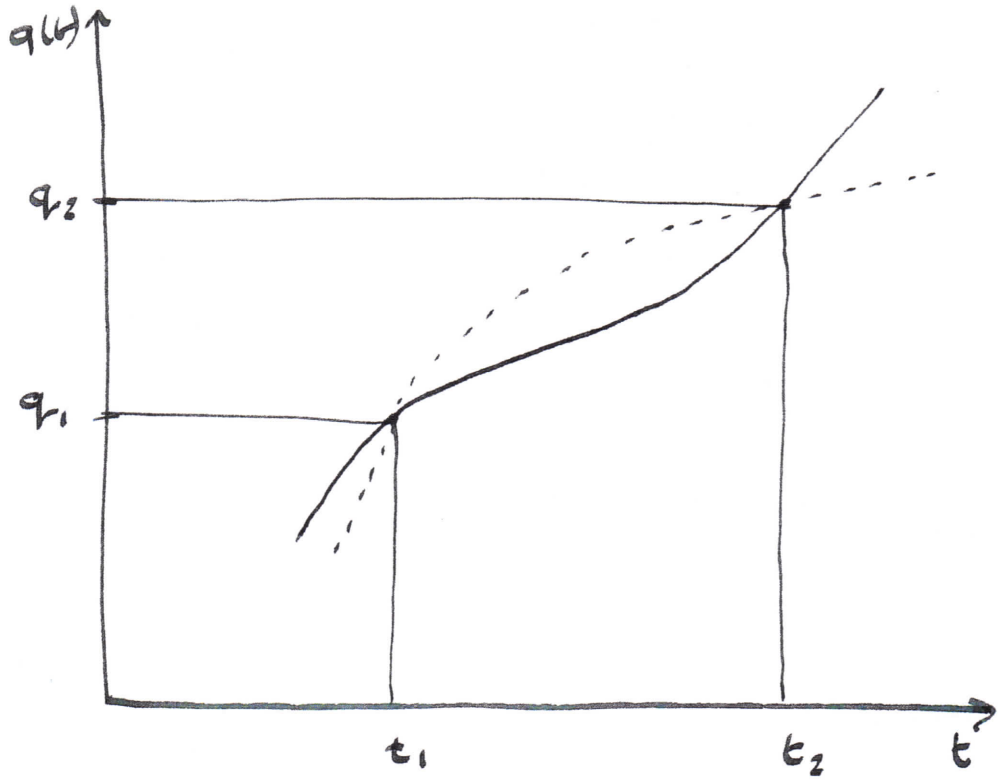
What do we mean by a field of conjugate points:

Imagine a string which can vibrate, each point along the string labelled by position x. There is a value $\phi(x)$ that can be attached to that string. It could be the transverse or longitudinal displacements or any other variable which can described a wave.

One can then proceed to form a quantum mechanical description by using the Lagrangian and canonical quantization.

The Lagrangian and the principle of least Action

Supposing a system is described by $q(t)$ and t we can express the state of that system as a point on a $q(t)$ t graph.



The curve on this graph describes what the system does, as a function of $q(t)$.

What describes or determines what route the system takes.

We can use the principle of least action to describe or find this curve. The curve it will take is the one for which the action $S(t_1, t_2)$ is a minimum.

For a conservative system the Action is described in terms of the Lagrangian

$$L = T - V \quad ; \quad S = \int_{t_1}^{t} L \, dt$$

We can use the calculus of variations to find the least Action Solutions.
We need to find the trajectory for which the integral of the Lagrangian (assumed only to depend on $(q(t), \dot{q}(t), t)$)

So

$$S = \int_{t_1}^{t_2} L\{q(t), \dot{q}(t), t\}\, dt$$

let $q(t)$ be the solution and

$$q(t) + \epsilon \phi(t)$$

a different path such that $\phi(t_1) = \phi(t_2) = 0$ and ϵ is a small quantity.

The condition that S is a minimin can be expressed as $\frac{dS}{d\epsilon} = 0$.

(If $q(t)$ is for at t between t_1 and t_2 the minimin then $dS/d\bar{\epsilon}^0$ will be a minimum).

$$S = \int_{t_1}^{t_2} L(q + \epsilon\phi,\ \dot{q} + \epsilon\dot{\phi},\ t)\, dt$$

$$\frac{dS}{d\epsilon} = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q}\phi + \frac{\partial L}{\partial \dot{q}}\dot{\phi}\right] dt = 0.$$

Integrating the second term by parts

$$\int_{t_1}^{t_2} \frac{\partial L}{\partial \dot{q}} \frac{d\phi}{dt}\, dt = \frac{\partial L}{\partial \dot{q}}\phi(t)\Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \phi(t) \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\, dt$$

$$\Big\| \quad 0$$

as $\phi(t_1), \phi(t_2) = 0$.

Thus $$\frac{dS}{d\epsilon} = \int_{t_1}^{t_2} \left[\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right] \phi(t)\, dt = 0.$$

This integral must be zero for arbitrary $\phi(t)$

Thus

$$\left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right) = 0.$$

Euler - Lagrange Equation

for several independent coordinates q_i ($i = 1, 2, \ldots n$) the generalization is simple.

$$S = \int_{t_1}^{t_2} L\{q_1, q_2 \ldots q_n, \dot{q}_1, \dot{q}_2 \ldots \dot{q}_n, t\}\, dt.$$

but this simply gives an Euler Lagrange equation for each set of variables.

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0.}$$

Hamiltonian's Equations

Hamilton exchanged the dependence on the velocities with a dependence on the generalized momenta and is expressed as the sum of the $T + V$ is energy.

$$H = p\dot{q} - L(q, \dot{q}, t)$$

in this term is $2T$

$$= 2T - T + V$$

$$= T + V.$$

$$dH = p\,d\dot{q} + \dot{q}\,dp - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial \dot{q}}d\dot{q} - \frac{\partial L}{\partial t}dt$$

Since $p = \frac{\partial L}{\partial \dot{q}}$ the 2nd and 4th terms cancel.

$$\therefore dH = \dot{q}\,dp - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial t}dt$$

But $\frac{\partial L}{\partial q} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{d}{dt}p = \dot{p}$

$$dH = \dot{q}\,dp - \dot{p}\,dq - \frac{\partial L}{\partial t}dt$$

$$\therefore \dot{q} = \frac{\partial H}{\partial p} \quad ; \quad \dot{p} = -\frac{\partial H}{\partial q} \quad ; \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}.$$

$$\therefore \frac{dH}{dt} = \dot{q}\frac{dp}{dt} - \dot{p}\frac{dq}{dt} - \frac{\partial L}{\partial t}$$

$$\therefore \frac{dH}{dt} = \frac{\partial H}{\partial t}$$

Thus if t does not appear explicitly in the Hamiltonian it is a constant of time.

So back to quantizing our simple string where we are considering every point on the string a set of independent variables

$$L(q, \dot{q}) = T - V$$

$$p_i \equiv \frac{\partial L}{\partial \dot{q}_i}$$

$$H(p, q) \equiv \sum_i p_i \dot{q}_i - L$$

We use the Heisenberg picture where state vectors stay constant and all time dependencies are carried by the operator.

Then we introduce equal-time commutation relations.

$$[q_i(t), p_j(t)] = i\delta_{ij} \quad ; [q_i(t), q_j(t)] = 0$$

$$[p_i(t), p_j(t)] = 0.$$

In the Heisenberg picture $O(t)$

$$-i\dot{O}(t) = [H, O]$$

The matrix element is still defined the same way.

$\langle \phi_1 | O(t) | \phi_2 \rangle$ except the states are independent of time and that is transferred to $O(t)$.

Thus we can define $\phi(x)$ as the displacement or as $\phi_i(x_i)$ for the ith material in the string.

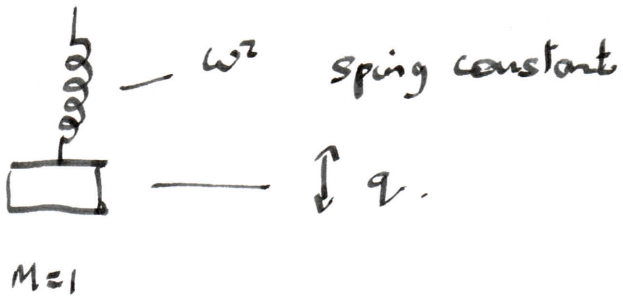
Or we can choose normal modes.

$$\phi(x) = \sum_p c_p e^{ipx} + \sum_p c_p^* e^{-ipx}$$

These normal modes would just be those harmonics one would expect from the boundary conditions and taking the Fourier transform.

So it becomes natural to take the amplitudes of the normal modes as the general coordinates. The energy in each normal mode can be thought of as due to a number of particles.

Harmonic Oscillator



$$L(q,\dot{q}) = T - V = \frac{1}{2}(\dot{q}^2 - \omega^2 q^2)$$

The Euler-Lagrange equation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} \rightarrow \ddot{q} = -\omega^2 q.$$

$$\frac{\partial L}{\partial \dot{q}} = \dot{q} \qquad \frac{\partial L}{\partial q} = -\omega^2 q$$

$$\therefore q(t) \propto e^{\pm i\omega t} \qquad q(t) = c\,e^{-i\omega t} + c^* e^{i\omega t}.$$

c: arbitrary complex constant.

$$p \equiv \frac{\partial L}{\partial \dot{q}} = \dot{q}$$

$$H(p,q) \equiv p\dot{q} - \frac{1}{2}(\dot{q}^2 - \omega^2 q^2)$$

$$= \dot{q}^2 - \frac{1}{2}(\dot{q}^2 - \omega^2 q^2) \Rightarrow$$

$$= \frac{1}{2}(p^2 + \omega^2 q^2)$$

if we impose the equal time commutation relations

$$[q(t), p(t)] = i \qquad [q(t), q(t)] = [p(t), p(t)] = 0$$

$$-i\dot{\theta} = [H, \theta]$$

$$-i\dot{q} = [H, q] = \frac{1}{2}[p^2 + \omega^2 q^2, q] = \frac{1}{2}[p^2, q]$$

Now,

$$[AB, C] = A[B, C] + [A, C]B.$$

easily proved

$$= -ip \qquad \therefore \dot{q} = p.$$

$$-i\dot{p} = [H, p] = \frac{1}{2}[p^2 + \omega^2 q^2, p]$$

$$= \frac{1}{2}[\omega^2 q^2, p]$$

$$= i\omega^2 q$$

$$\therefore \dot{p} = -\omega^2 q.$$

Define

$$a(t) \equiv \frac{1}{\sqrt{2}}\left(\sqrt{\omega}\, q + i\frac{p}{\sqrt{\omega}}\right)$$

$$a^{+}(t) = \frac{1}{\sqrt{2}}\left(\sqrt{\omega}\, q - i\frac{p}{\sqrt{\omega}}\right)$$

Note $a(t)$ is not hermitian.

$$\left[a(t), a^{+}(t)\right] = \frac{1}{2}\left[\sqrt{\omega}\, q + i\frac{p}{\sqrt{\omega}}, \sqrt{\omega}\, q - i\frac{p}{\sqrt{\omega}}\right]$$

$$= \frac{1}{2}\left(i\underbrace{[p,q]}_{-i} - i\underbrace{[q,p]}_{i}\right) = 1$$

The Hamiltonian can be written in terms of these terms $a(t)$ $a^\dagger(t)$ as follows:-

$$a(t)\,a^\dagger(t) + a^\dagger(t)\,a(t)$$

$$= \frac{1}{2}\left\{\left(\sqrt{\omega}q + i\frac{p}{\sqrt{\omega}}\right)\left(\sqrt{\omega}q - i\frac{p}{\sqrt{\omega}}\right) + \left(\sqrt{\omega}q - i\frac{p}{\sqrt{\omega}}\right)\left(\sqrt{\omega}q + i\frac{p}{\sqrt{\omega}}\right)\right\}$$

$$= \frac{1}{2}\left(2\omega q^2 + 2\frac{p^2}{\omega}\right) = \frac{1}{\omega}\underbrace{\left(\omega^2 q^2 + p^2\right)}_{2H}$$

Thus

$$H = \frac{\omega}{2}\left(\underbrace{a(t)\,a^\dagger(t)}_{1 + a^\dagger(t)a(t)} + a^\dagger(t)\,a(t)\right)$$

$$\boxed{H = \omega\left(a^\dagger(t)\,a(t) + \frac{1}{2}\right)}$$

The time dependence of $a(t)$

$$+i\,\dot{a}(t) = [a(t), H] = [a(t),\ a^\dagger(t)\,a(t)]$$

$$= \omega\underbrace{[a^\dagger(t), a(t)]}_{-1}\,a(t).$$

$$\therefore \quad \dot{a}(t) = -i\omega a(t)$$

This has general solution

$$\boxed{a(t) = a e^{-i\omega t} \quad \text{with} \quad a \equiv a(0).}$$

The symbol $a \equiv a(0)$

We can define the commutation relations set in terms of a, a^+.

$$[a, a^+] = 1$$

$$H = \omega\left(N + \tfrac{1}{2}\right) \qquad N \equiv a^+ a.$$

where N is called the NUMBER OPERATOR.

The Number Operator

$$[A, BC] = [A, B]C + B[A, C]$$

$$[a^+, N] = [a^+, a^+ a] = \underbrace{[a^+, a^+]}_{0} a + a^+ \underbrace{[a^+, a]}_{-1}$$

$$= -a^+$$

$$\therefore \quad N a^+ = a^+ (N+1)$$

(In other words if you operate by N^+ first then you get a $N+1$ factor.)

$$[a, N] = [a, a^\dagger a] = a^\dagger \underbrace{[a, a]}_{0} + \underbrace{[a, a^\dagger]}_{+1} a$$

$$= a .$$

$$\therefore \quad Na = a(N-1) .$$

So what does this imply?

Let $|n\rangle$ be an eigenstate of N eigenvalue n.

$$\therefore \quad N|n\rangle = n|n\rangle$$

note that since N is hermitian n is real.

$$\left(N = a^\dagger a \qquad N^\dagger = (a^\dagger a)^\dagger = a^\dagger a = N\right).$$

So
$$N a^\dagger |n\rangle = a^\dagger (N+1)|n\rangle$$
$$\Rightarrow N(a^\dagger |n\rangle) = n+1\,(a^\dagger |n\rangle)$$
} Thus $a^\dagger$ can be thought of as a creation operator

$$N a|n\rangle = a(n-1)|n\rangle$$
$$N(a|n\rangle) = (n-1)(a|n\rangle)$$
} a an annihilation operator.

$n \geq 0$

Let $\langle n|N|n\rangle = \langle n|a^\dagger a|n\rangle = |a|n\rangle|^2 \geq 0.$

But $\langle n|N|n\rangle = n\underbrace{\langle n|n\rangle}_{\geq 0}$.

Thus $n\underbrace{\langle n|n\rangle}_{>0} = |a(n)|^2 \geq 0$.

Thus $n > 0$.

Thus the number of states n must be positive.

n = integer

Now if n were a fractional value for a given eigenvalue. It would be possible to apply $a^k|n\rangle$ until we reached a state where

$n < 0$.

The only way to avoid this is for every n to be integer. At some point $a^k(n)$

$= |0\rangle$

So what happens if you attempt to go past that point

Consider $a|0\rangle$

$$|a|0\rangle|^2 = \langle 0|a^\dagger a|0\rangle = 0.\langle 0|0\rangle$$
$$= 0.$$

This implies $a|0\rangle = 0$ and. so if the original state $|n\rangle$ is integer applying a will always result in $a|0\rangle = 0$ and the negative states are not accessible.

All eigenstate can be formed by:-

$$|n\rangle \equiv \frac{1}{\sqrt{n!}} a^{\dagger n} |0\rangle$$

Proof:

we have shown that

$$N(a^\dagger|n\rangle) = (n+1)\underbrace{(a^\dagger|n\rangle)}_{\alpha|n+1\rangle}$$

$\therefore a^\dagger|n\rangle = \alpha|n+1\rangle$. α = arbitrary factor

But $|a^\dagger|n\rangle|^2 = \langle n|aa^\dagger|n\rangle$ $\quad [a, a^\dagger] = 1$

$aa^\dagger = 1 + N$,

$= \langle n|1+N|n\rangle$

$= 1+n\langle n|n\rangle = 1+n$

$\therefore |a^\dagger|n\rangle|^2 = 1 + n$

$$\therefore \quad \boxed{a^\dagger |n\rangle = \sqrt{1+n}\ |n+1\rangle}$$

Also

$$N(a|n\rangle) = (n-1)\underbrace{(a|n\rangle)}_{|n+1\rangle}$$

$$\therefore \quad a|n\rangle = \alpha |n+1\rangle$$

But $$|a|n\rangle|^2 = \langle n|a^\dagger a|n\rangle$$

$$= n\langle n|n\rangle$$

$$\therefore \quad \boxed{a|n\rangle = \sqrt{n}\ |n-1\rangle}$$

So if we consider starting from $|0\rangle$

$$a^\dagger|0\rangle = \sqrt{n+1}\ |n+1\rangle$$

$$= \sqrt{1}\,|1\rangle$$

$$\sqrt{1}\,a^\dagger|1\rangle = \sqrt{2}\,|2\rangle$$

$$\sqrt{1}\sqrt{2}\,a^\dagger|2\rangle = \sqrt{3}\,|3\rangle \qquad \therefore \sqrt{n!}\,a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

$\therefore$ if we want a normalized state

$$|n\rangle = \frac{1}{\sqrt{n!}}\,a^{\dagger n}|0\rangle$$

Fermionic Oscillator

We know that for fermions the state can only be occupied by one or zero particles. This is achieved by replacing the commutator relationships with anticommutator relations.

$$\{a, a^+\} = 1 \qquad \text{instead of} \quad [a, a^+] = 1.$$

$$\{a, a\} = \{a^+, a^+\} = 0$$

instead of $[a,a] = [a^+, a^+] = 0$.

But $\{a,a\} = aa + aa = 2aa = 0.$

$\{a^+, a^+\} = a^+a^+ + a^+a^+ = 2a^+a^+ = 0$

$\therefore \quad a^2 = a^{+2} = 0.$

Definition of $N = a^+a$ as before

$$N^2 = a^+a\underbrace{a^+a}_{1-aa^+} = a^+a(1-aa^+)$$

$$= a^+a - a^+\underbrace{aa}_{0}a^+$$

$$= N$$

$\therefore \quad N(N-1) = 0 \quad \Rightarrow \quad N = 0, 1.$ only.

If we define

$$\langle 0|0\rangle = 1.$$

Applying the creation operator $a^\dagger$

$$N a^\dagger = \underbrace{a^\dagger a}_{1 - a a^\dagger}\, a^\dagger = a^\dagger - a \underbrace{a^{\dagger 2}}_{0} = a^\dagger .$$

Thus $$N(a^\dagger|0\rangle) = (a^\dagger|0\rangle)$$

$$\therefore \quad N(a^\dagger|0\rangle) = 1.(a^\dagger|0\rangle).$$

$$\therefore \quad |1\rangle = a^\dagger|0\rangle$$

$$\langle 1|1\rangle = \underbrace{\langle 0|a a^\dagger|0\rangle}_{1 - a^\dagger a} = \langle 0|1 - \underbrace{N}_{0}|0\rangle = \langle 0|0\rangle = 1$$

The annihilation operator

$$a|1\rangle = a a^\dagger|0\rangle = (1-N)|0\rangle = |0\rangle .$$

Summary

$$a^\dagger|1\rangle = a^{\dagger 2}|0\rangle = 0.$$

$$a|0\rangle = a^2|0\rangle = 0.$$

The hamiltonian is proportional to the number of particles up to a constant which can be ignored.

$$\boxed{H = \omega a^{\dagger} a}$$