Wick's Theorem

This is the relationship between the time ordered products and the normal ordered products and for two fields it is just what we have just derived.

$$\mathcal{T}[\phi(x_1)\,\phi(x_2)] = N[\phi(x_1)\,\phi(x_2)] + \underbrace{\phi(x_1)\,\phi(x_2)}$$

if the fields correspond to the same particle

$$\underbrace{\phi(x_1)\,\phi(x_2)} = \langle 0|\mathcal{T}[\phi(x_1)\phi(x_2)]|0\rangle$$

$$= i\Delta_F(x_1 - x_2)$$

$$\phi(x_1)\,\phi^{+}(x_2) = i\Delta_F(x_1 - x_2)$$

$= \phi^{+}(x_2)\,\phi(x_1)$ - for bosons.

$$\psi_\alpha(x_1)\,\overline{\psi}_\beta(x_2) = -\overline{\psi}_\beta(x_2)\,\psi_\alpha(x_1)$$ - for fermions

$$= i\,S_{F\alpha\beta}(x_1 - x_2)$$

where $\Delta_F(x_1 - x_2)$ and $S_{F\alpha\beta}(x_1 - x_2)$ are Feynman Propagators. We will discuss these more later but they refer to a particle created at x_2 and destroyed at x_1.

Wick's Theorem can now be generalized to any number of fields (its proof is by induction and not very enlightening so will be omitted).

$$\mathcal{T}\left[\phi_1(x_1)\ldots\ldots\ldots\phi_n(x_n)\right]$$

$$= N\left[\phi_1 \quad \ldots\ldots \quad \phi_n\right]$$

$$+ \quad \phi_1\phi_2\, N\left[\phi_3\ldots\phi_n\right] + \phi_1\phi_3\, N\left[\phi_2\ldots\phi_n\right] + \ldots\ldots\ldots$$

$$+ \quad \underbrace{\phi_1\phi_2}\ \underbrace{\phi_3\phi_4}\ N\left[\phi_5\ldots\phi_n\right] + \quad \ldots\ldots\ldots$$

$\vdots$

$$+ \begin{cases} \underbrace{\phi_1\phi_2}\ \underbrace{\phi_3\phi_4}\ldots\ldots \underbrace{\phi_{n-1}\phi_n} + \ldots\ldots\ldots & \text{if } n \text{ even} \\ (\underbrace{\phi_1\ \phi_2}\ \underbrace{\phi_3\ \phi_4}\ldots\ldots \phi_{n-2}\phi_{n-1})\,\phi_n + \ldots\ldots & \text{if } n \text{ odd} \end{cases}$$

where ..... implied " all possible permutations".

Below is the Wick's expansion of 4 fields.

$$T[\phi_1 \phi_2 \phi_3 \phi_4] =$$

$$N[\phi_1 \phi_2 \phi_3 \phi_4]$$

$$+ N[\phi_1 \phi_2 \phi_3 \phi_4] + N[\phi_1 \phi_2 \phi_3 \phi_4]$$

$$+ N[\phi_1 \phi_2 \phi_3 \phi_4] + N[\phi_1 \phi_2 \phi_3 \phi_4]$$

$$+ N[\phi_1 \phi_2 \phi_3 \phi_4] + N[\phi_1 \phi_2 \phi_3 \phi_4]$$

$$+ N[\phi_1 \phi_2 \phi_3 \phi_4]$$

The interpretation of these wick contractions $\phi(x_1)\phi(x_2)$ is that this gives a propagator (virtual particle) between x_1 an x_2.

we will investigate this more in the following pages.

The Feynman Propagator $\left(\underbrace{\phi(x) \quad \phi(y)}\right)$

We found that

$$\underbrace{\phi(x) \quad \phi(y)} = \langle 0 | \mathcal{T}[\, \phi(x)\, \phi(y)] | 0 \rangle$$

but since

$$\phi(x)\,\phi(y) = N[\phi(x)\,\phi(y)] + [\phi^+(x), \phi^-(y)]$$

where

$$\phi(x) = \phi^+(x) + \phi^-(x)$$

and

$$\phi^+(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_p}}\, a_p\, e^{-i\underline{p}\cdot\underline{x}}$$

$$\phi^-(x) = \int \frac{d^3p}{(2\pi)^3 \sqrt{2E_p}}\, a_p^+\, e^{i\underline{p}\cdot\underline{x}}$$

so $\underbrace{\phi(x) \quad \phi(y)} = \langle 0 | \mathcal{T}[\phi^+(x), \phi^-(y)] | 0 \rangle$

because $\langle 0 | N[\quad] | 0 \rangle = 0$.

This can be written

$$\langle 0 | \mathcal{T}[\phi^+(x), \phi^-(y)] | 0 \rangle =$$

$$\langle 0 |\, \theta(x^0 - y^0)[\phi^+(x)\, \phi^-(y)] +$$

$$\theta(y^0 - x^0)[\phi^+(y)\, \phi^-(y)]\, | 0 \rangle$$

since we can just swap $x \leftrightarrow y$ for bosons.

$$\phi(x)\,\phi(y) = \phi(y)\,\phi(x)$$

and

$$\phi(x)\,\phi(y) = N[\phi(x)\,\phi(y)] + [\phi^+(x), \phi^-(y)]$$
$$\phi(y)\,\phi(x) = N[\phi(y)\,\phi(x)] + [\phi^+(y), \phi^-(x)]$$

but using the definitions above for $\phi^+(x)$ and $\phi^-(x)$ etc.

we can write

$$\langle 0 | \mathcal{T}[\phi^+(x), \phi^-(y)] | 0 \rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big(\theta(x^0 - y^0)\, e^{-ip.(x-y)} [a_p a_p^+] + \theta(y^0 - x^0)\, e^{ip.(x-y)} [a_p a_p^+] | 0 \rangle$$

But $[a_p a_p^+] = 1$

so $\phi(x)\,\phi(y) \equiv \langle 0 | \mathcal{T}[\phi^+(x)\,\phi^-(y)] | 0 \rangle$

can be written

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \Big(\theta(x^0 - y^0)\, e^{-ip.(x-y)} + \theta(y^0 - x^0)\, e^{ip.(x-y)} \Big)$$

Expressing the Heavyside step function θ as an integral

Consider the following

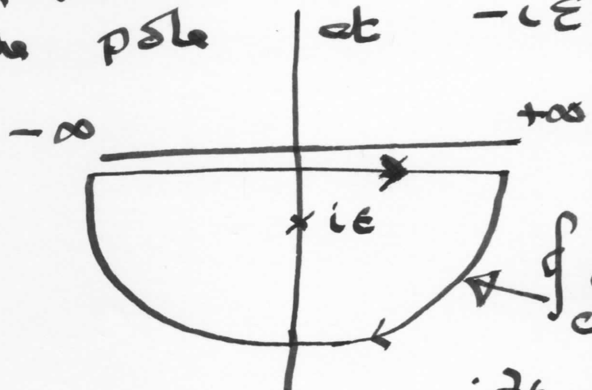
$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} d\alpha \, \frac{e^{-i\alpha t}}{\alpha + i\epsilon} = -2\pi i \, \theta(t)$$

re expressing this as the contour in the complex plane.

$$\oint dz \, \frac{e^{-izt}}{z+i\epsilon} = \int_{-\infty}^{\infty} dz \, \frac{e^{-izt}}{z+i\epsilon} + \int_C dz \, \frac{e^{-izt}}{z+i\epsilon}$$

we can use the above to calculate the integral along the real axis if the closing contour C contributes nothing.

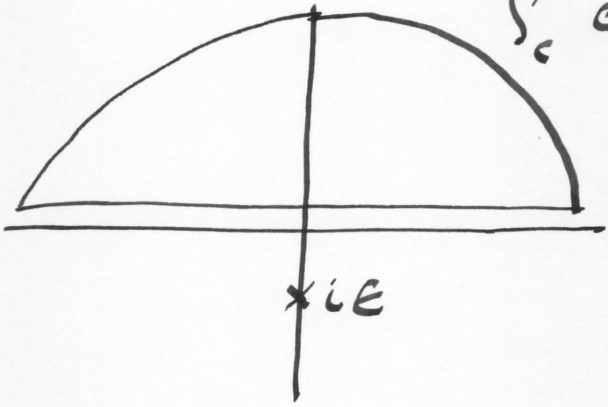
So if $\underline{\underline{z > 0}}$ we must chose the contour in the lower half plane which encloses the pole at $-i\epsilon$.



$\int_C$ contributes nothing. $e^{-izt} \xrightarrow[z \to \infty]{} 0$.

So $$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} dz \, \frac{e^{-izt}}{z+i\epsilon} = -2\pi i$$

And if $z < 0$ we must close the contour in the upper half plane, which does not enclose the pole



$\int_c$ contributes nothing

$e^{iZt} \to 0$

$z \to \infty$

$$\text{So} \lim_{\epsilon \to 0} \int dz \, \frac{e^{-iZt}}{z + i\epsilon} = 0$$

Thus the Heavyside step function as a function of t can be expressed as an integral along the real axis in α but also as a function of t.

$$-2\pi i \; \theta(t) = \lim_{\epsilon \to 0} \int_{-\infty}^{\infty} d\alpha \, \frac{e^{-i\alpha t}}{\alpha + i\epsilon} .$$

We will make use of this when considering the Feynman propagator.

It is more useful to express the propagator in the form

$$\underbrace{\phi(x)\,\phi(y)} = \int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2 - m^2 + i\epsilon}\,e^{-ip(x-y)}$$

where $\epsilon \longrightarrow 0^+$

To prove the equivalence of this form and the previous expression the above can be rewritten not as a pole in p^2 but as a pole in the single variable p^0.

$$\Rightarrow \int \frac{d^3p}{(2\pi)^3}\, e^{-i\underline{p}\cdot(\underline{x}-\underline{y})} \int_{-\infty}^{\infty} \frac{dp^0}{(2\pi)}\,\frac{i}{(p^0)^2 - E_p^2 + i\epsilon}\,e^{-ip^0(x^0-y^0)}$$

where $E_p^2 = p^2 + m^2$

So where are the poles in the p^0?

$$(p^0)^2 = E_p^2 - i\epsilon$$

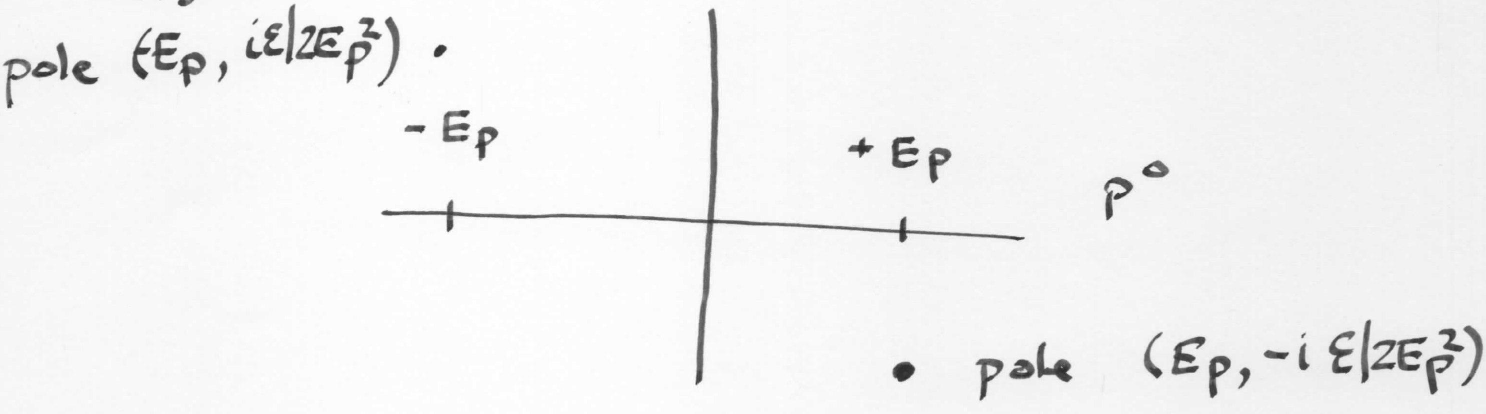
$$= E_p^2\,(1 - i\epsilon/E_p^2)$$

$$\therefore\ p^0 = \pm E_p\,(1 - i\epsilon/E_p^2)^{1/2}$$

$$\simeq \pm E_p\,(1 - i\epsilon/2E_p^2)$$

Thus there will be two poles at $\pm E_p$ on the real axis. However the pole at $+E_p$ will be displaced by $-i\varepsilon/2E_p^2)$ and the one at $-E_p$ will be displaced to be slightly above the real axis at $+i\varepsilon/2E_p^2$.

The poles can be represented as



So using the theorem of residues

$$\oint \frac{f(z)}{(z - z_0)}\, dz = 2\pi i\, f(z_0)$$

so
$$\int_{-\infty}^{\infty} \frac{dp^0}{(2\pi)} \frac{i}{(p^0)^2 - E_p^2 + i\varepsilon} e^{-ip^0(x^0 - y^0)}$$

This can be rewritten as two poles $E_p - i\varepsilon/2E_p^2$ and $-E_p + i\varepsilon/2E_p^2$ which in the limit as $\varepsilon \to 0^+$ becomes $\pm E_p$

so the above can be written

$$\oint \frac{dp^0}{(2\pi)} \frac{i}{(p^0 + E_p)(p^0 - E_p)} e^{-ip^0(x^0 - y^0)}$$

So if we consider the pole at $\simeq E_p$

$$f(E_p) = \frac{i e^{-ip^0(x^0 - y^0)}}{2\pi\,(p^0 + E_p)}$$

so
$$2\pi i f(E_p) = \frac{i}{2\pi}\; -2\pi i\; \frac{e^{-iE_p(x_0 - y_0)}}{2E_p}$$

$$= \frac{1}{2E_p} e^{-iE_p.(x^0 - y^0)}$$

and at $-E_p$
$$= -\frac{1}{2E_p} e^{+iE_p(x^0 - y^0)}$$

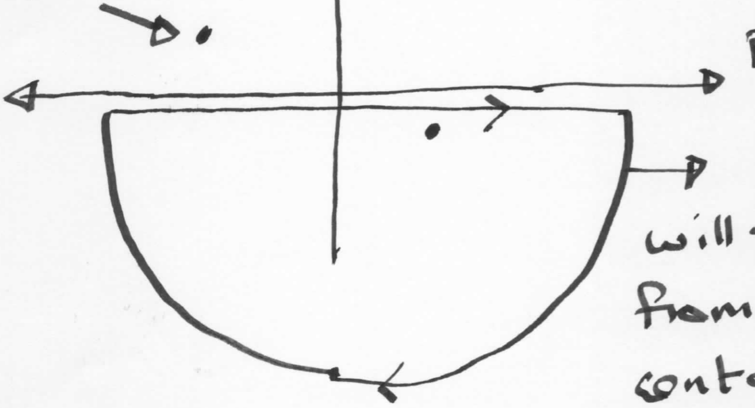
The case where $x_0 - y_0 > 0$

The integral being considered is

$$\int \frac{d\underline{p}^3}{(2\pi)^3} e^{i\underline{p}\cdot(\underline{x}-\underline{y})} \int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \frac{i}{(p^0)^2 - E_p^2 + i\epsilon} e^{-ip^0(x^0-y^0)}$$

The $e^{-ip^0(x^0-y^0)}$ term will allow a contour to evaluate the integral along the real axis if it is chosen to be in the lower half of the complex plane

poles



the term $e^{-ip^0(x^0-y^0)}$ will ensure the contribution from this part of the contour is ϕ.

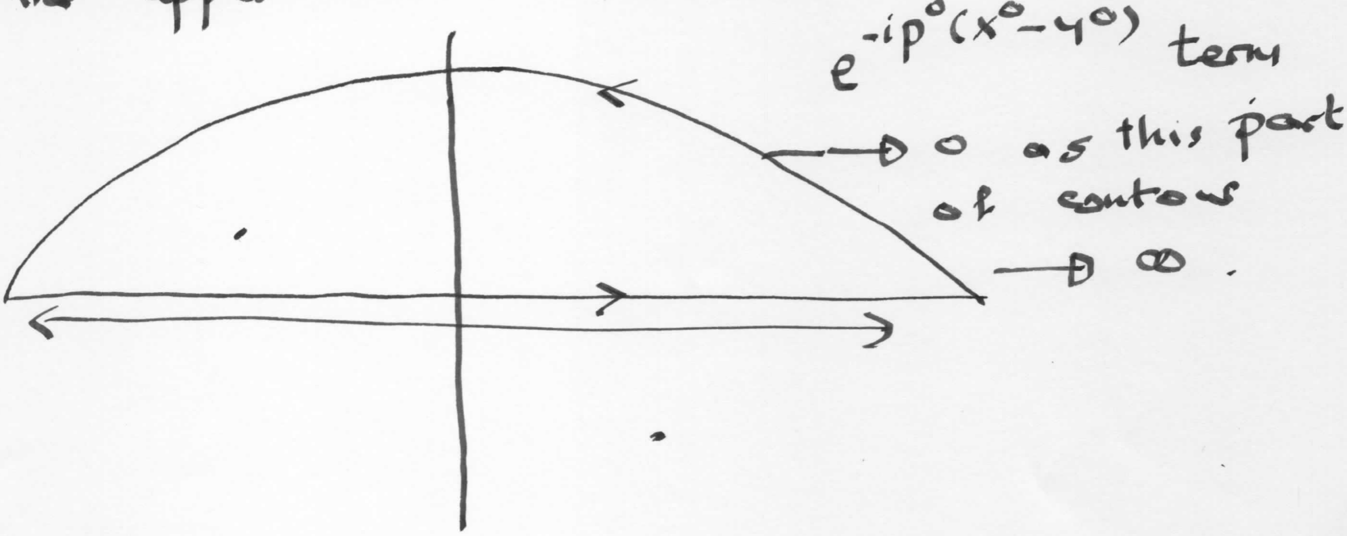
Only the one pole at $E_p - i\epsilon/2E_p^2$ is enclosed in a clockwise direction : $\int \frac{f(z)}{z - z_0} dz = -2\pi i f(z_0)$.

Thus the contour evaluates the time integral to be $\frac{i}{2\pi}(-2\pi i)\frac{1}{2E_p} e^{-iE_p(x^0-y^0)}$

$$= \frac{1}{2E_p} e^{-iE_p(x^0-y^0)}$$

The case where

In this case the $e^{-ip^0(x^0-y^0)}$ term will allow the contour to evaluate the integral along the real axis only if the contour in the upper half is chosen



This time only the pole at $-E_p + i\varepsilon/2E_p^2$ is enclosed and $\oint \frac{f(z)}{z-z_0}\,dz = +2\pi i f(z_0)$

so the time integral along the real axis

$$\int_{-\infty}^{\infty} \frac{dp^0}{2\pi} \frac{i}{(p^0)^2 - E_p^2 + i\epsilon} e^{-ip^0(x^0-y^0)}$$

with $p^0 = -E_p$

becomes $\frac{i}{2\pi} 2\pi i \frac{e^{iE_p(x^0-y^0)}}{-2E_p} = \frac{1}{2E_p} e^{iE_p(x^0-y^0)}$

So the term

$$\phi(x)\ \phi(y) = \int \frac{d^3p}{(2\pi)^3}\,\frac{1}{2E_p}\left(\theta(x^0-y^0)\,e^{-ip(x-y)}+\theta(y^0-x^0)\,e^{ip(x-y)}\right)$$

is reproduces by

$$\int\frac{d^3p}{(2\pi)^3}\, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\int_{-\infty}^{\infty}\frac{dp^0}{2\pi}\,\frac{i\; e^{-ip^0(x^0-y^0)}}{(p^0)^2-E_p^2+i\epsilon}$$

which we have shown to be

$$\int\frac{d^3p}{(2\pi)^3}\, e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}\,\frac{1}{2E_p}\left(e^{-iE_p(x-y)}\,\theta(x^0-y^0) + e^{iE_p(x-y)}\,\theta(y^0-x^0)\right)$$

where for the second term, since

$$p(x-y) \equiv p^\mu(x-y)_\mu = p^0(x^0-y^0) - \mathbf{p}\cdot(\mathbf{x}-\mathbf{y})$$

we must change the variable of integration from $\mathbf{p}\to-\mathbf{p}$; but the value of the integral is the same.

so we have shown

$$\phi(x)\;\phi(y) \;=\; \int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-m^2+i\varepsilon}\,e^{-ip(x-y)}$$

This is also called $D(x-y)$ in some literature and $i\,\Delta_F(x-y)$ in others.

if
$$D(x-y) \;=\; \int \frac{d^4p}{(2\pi)^4}\,\frac{i}{p^2-m^2+i\varepsilon}\,e^{-ip(x-y)}$$

the by definition
$$\tilde{D}(p) \;=\; \frac{i}{p^2-m^2+i\varepsilon}$$

which is the fourier transform of the propagator in p space.

It is also easy to see that the Feynmann propagator is just the Green's function of the operator $\Box + m^2$

So
$$(\Box_x + m^2)\; D(x-y)$$

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - m^2 + i\varepsilon} (-p^2 + m^2)\, e^{-ip(x-y)}$$

$$= -\int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)}$$

$$= -\delta^4(x-y)$$

This holds independently of which contour is used.

In fact there are four different prescriptions for displacing the poles and each solutions satisfies different boundary conditions.

Wick expansion to Feynman Diagrams

We are going to use the Wick expansion to calculate the S-matrix elements involving scalars and spinors.

use Yukawa interaction

$$\mathcal{L}_{int} = -h\bar{\psi}\psi\phi$$

Let particle represented by field ϕ be B and ψ are electrons.

$B_{mass} = M > 2m_e$, so B can decay to an electron positron pair.

so $B(k) \longrightarrow e^-(p) + e^+(p)$

So $\mathcal{H}_I = h:\bar{\psi}\psi\phi:$ — linear in each field.

So look at the first (linear term) in the S-matrix.

$$S^{(1)} = \mathcal{T}\left[-i\int d^4x\, \mathcal{H}_I(x)\right]$$

$$= -ih\int d^4x :\bar{\psi}\psi\phi:$$

we can omit the time ordering as there is just one space-time point, the interaction point of $\psi\bar{\psi}\phi$. So Wicks theorem is not needed here.

Let us now expand in terms of creation and annihilation terms.

$$S^{(1)} = -ih \int d^4x : (\bar{\psi}_+ + \bar{\psi}_-)(\psi_+ + \psi_-)(\phi_+ + \phi_-)$$

annihilation e^+
etc.

Consider now the matrix element for

$B \to e^+ e^-$.

So $\phi_- |B\rangle$ — will create another B particle in the initial state and this can never match the final state so this term must vanish.

But $\phi_+ |B\rangle \to |0\rangle$

and then we can have $\bar{\psi}_- \psi_- |0\rangle$ create $e^+ e^-$ pair!

So for the possible eight terms only

$$-i\hbar \int d^4x \, \bar{\Psi}_- \Psi_- \phi_+ \quad \text{will contribute to}$$

$$B \longrightarrow e^+ e^- .$$

This is already normally ordered and occurs at one space time point x and can be represented as

This is trivial Feynman diagram.

The second order term — cannot contribute this contains six field operators. We only need three to annihilate and create what we need, leaving 3 more. Since this is an odd number we cannot arrive at our final state.

The first non-trivial corrections come in $S^{(3)}$

$$S^{(3)} = \frac{(-ih)^3}{3!}\int d^4x_1 \int d^4x_2 \int d^4x_3$$

$$T\left[:(\bar\psi\psi\phi)_{x_1}: \; :(\bar\psi\psi\phi)_{x_2}: \; :(\bar\psi\psi\phi)_{x_3}:\right]$$

But Wick's theorem says we can contract away the six unwanted field operators

$$T\left[:(\bar\psi\psi\phi)_{x_1}: \; :(\bar\psi\psi\phi)_{x_2}: \; :(\bar\psi\psi\phi)_{x_3}:\right]$$

$$= \; :(\bar\psi\psi\phi)_{x_1}(\bar\psi\psi\phi)_{x_2}(\bar\psi\psi\phi)_{x_3}:$$

$$+ \; :(\bar\psi\psi\phi)_{x_1}(\bar\psi\psi\phi)_{x_2}(\bar\psi\psi\phi)_{x_3}:$$

$$+ \; :(\bar\psi\psi\phi)_{x_1}(\bar\psi\psi\phi)_{x_2}(\bar\psi\psi\phi)_{x_3}:$$

$$+ \; :(\bar\psi\psi\phi)_{x_1}(\bar\psi\psi\phi)_{x_2}(\bar\psi\psi\phi)_{x_3}:$$

$$+ \; :(\psi\psi\phi)_{x_1}(\bar\psi\psi\phi)_{x_2}(\bar\psi\psi\phi)_{x_3}:$$

We can represent all these terms a Feynman diagrams (47)

Lets expand the first term

↓ fermions

$$- \underbrace{\phi(x_2)\,\phi(x_3)}\;\underbrace{\bar\psi_\alpha(x_1)\,\psi_\beta(x_2)}\;\underbrace{\psi_\alpha(x_1)\,\bar\psi_\beta(x_3)}$$
$$N[\psi_\beta(x_2)\,\psi_\gamma(x_3)\,\phi(x_1)]$$

re write as

$$+ \underbrace{\phi(x_2)\,\phi(x_3)}\;\underbrace{\psi_\beta(x_2)\,\bar\psi_\alpha(x_1)}\;\underbrace{\psi_\alpha(x_1)\,\bar\psi_\gamma(x_3)}$$
$$N[\bar\psi_\beta(x_2)\,\psi_\gamma(x_3)\,\phi(x_1)]$$

In the normal ordered factor the only term that gives a non-zero matrix element

$$\bar\psi^{\beta}_{-}(x_2)\;\psi^{\gamma}_{-}(x_3)\;\phi_{+}(x_1)$$

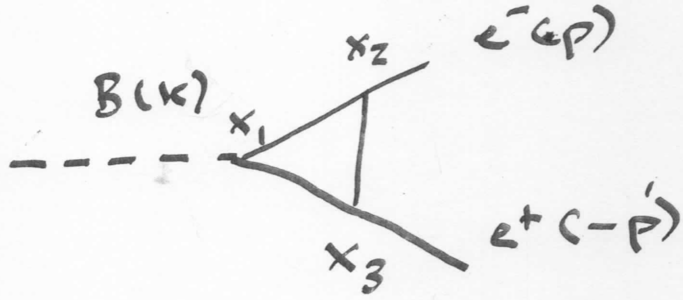
create e^+ create e^- annihilate B

The B annihilated at x_1

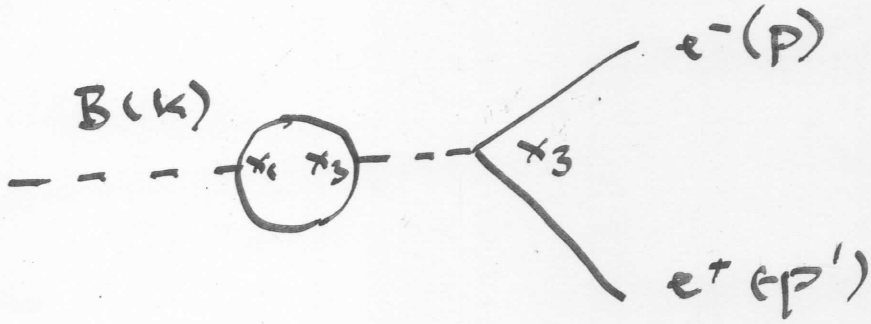
e^- created at x_2

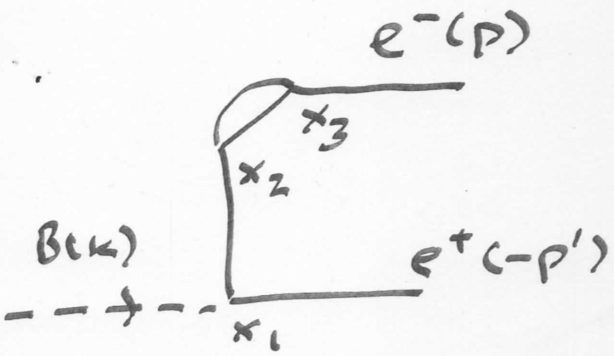
e^+ " at x_3.

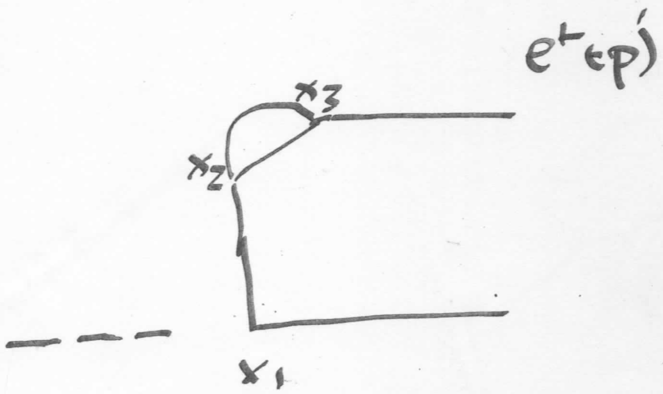
So this term becomes:—



the next three terms:—







Note that this is one common description of the S-Matrix term in the literature.

Note the following.

> The "interaction" term $\mathcal{H}_I$ is left to create the initial and final states from the vacuum.

> The higher order terms from this matrix element based on the Yukawa interaction is <u>only</u> capable of producing higher order terms based on that interaction — whereas real state particles can have those state available from all their possible interactions.

Some treatments of the S matrix element $\langle f|S|i\rangle$ explicitly create $|i\rangle = a^{+}|0\rangle$ out with of the $\mathcal{H}_I$ fields $\langle f| = \langle 0|a$, etc. This seems to me to give a more flexible approach, but does mean you have to be able to interpret

$$\underbrace{\phi(z)\;a^{+}}_{} = \frac{1}{(2\pi)^{3/2}}\frac{1}{(2E_p)^{1/2}}\,e^{-ip.z}$$

proof

$$\langle 0|\,\phi^{+}(z)\,a_p^{+}\,|0\rangle = \int\frac{d^3q}{(2\pi)^{3/2}}\frac{1}{(2E_q)^{1/2}}\langle 0|a_q e^{-iq.z} + a_q^{+}e^{iq.z})\,a_p^{+}|0\rangle$$

$$= \quad '' \quad \langle 0|(a_q e^{-iq.z} + a_q^{+}e^{iq.z})|p\rangle$$

$$= \quad '' \quad e^{-iq.z}\,\delta(q-p)$$

$$= \frac{1}{(2\pi)^{3/2}}\frac{1}{(2E_p)^{3/2}}\,e^{-ip.z}$$

Remember that QED is described by interactions from $\mathcal{H}_I = \bar{\psi} \not{A} \psi$ for photons and leptons $\phi^{+} A \phi$ for photons and scalar particles.