The S-Matrix Expansion

In our consideration of Q.F.T so far we have considered only "free" Lagrangian densities with no interacting fields.

From Q.E.D.

The free-field Lagrangian density is

$$\mathcal{L}_0 = N[\underbrace{\bar{\Psi}(x)(i\gamma^\mu\partial_\mu - m)\Psi(x)}_{\text{free lepton Lagrangian density}} - \tfrac{1}{2}\underbrace{(\partial_\nu A_\mu(x))(\partial^\nu A^\mu(x))}_{\text{free photon Lagrangian density}}]$$

the interaction Lagrangian is:

$$\mathcal{L}_I = N[-s^\mu(x) A_\mu(x)]$$

$$= N[e\bar{\Psi}(x)\gamma^\mu\Psi(x) A_\mu]$$

$$= N[e\bar{\Psi}(x) \not{A}(x) \Psi(x)]$$

These are written in normal ordering (all annihilation to the right of creation) and since $a|0\rangle = \langle 0|a^\dagger = 0$ this ensures that the vacuum expectation value for $\langle 0|a^\dagger \ldots\ldots a|0\rangle$ must always be zero.

Since $i\hbar \frac{d}{dt} |A,t\rangle_I = H_I^I |A,t\rangle_I$

$$= U_0^{+} H_I U_0 |A,t\rangle_I$$

If $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_I$

then $\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I$

If the $\mathcal{L}_I$ does not contain any derivatives of then the canonically conjugate fields or otherwise known as conjugate momenta are the same for $\mathcal{L}$ as $\mathcal{L}_0$

ie $\pi = \frac{\partial \mathcal{L}}{\partial \dot{q}} = \frac{\partial \mathcal{L}_0}{\partial \dot{q}}$

Since $\mathcal{H} = \sum \pi \dot{q} - \mathcal{L}$

$$\mathcal{H}_0 + \mathcal{H}_I = \sum \pi \dot{q} - \mathcal{L}_I - \mathcal{L}_0$$

it would be natural to make

which is same for $\mathcal{L}_0$ as $\mathcal{L}$.

$$\mathcal{H}_0 = \sum \pi \dot{q} - \mathcal{L}_0.$$

$$\mathcal{H}_I = \mathcal{L}_I$$

We showed for $H \Leftrightarrow S$ that operators that are related by a unitary transformation have the same commutation relations. They must therefore have the same equations of motion, except of course the operator is the interaction (IP) operator. $O^I(t)$.

Thus the IP states have the same solutions as before except $\mathcal{H} \rightarrow \mathcal{H}_I$.

The I.P. is defined as. (now with $\hbar \to 1$)

$$i \frac{d}{dt} |A, t\rangle_I = H_I(t) \, | A, t\rangle_I$$

or rewritten as

$$i \frac{d}{dt} |\phi(t)\rangle = H_I(t) |\phi(t)\rangle$$

where of course.

$$H_I(t) = e^{iH_0(t-t_0)} H_I^S e^{-iH_0(t-t_0)}$$

Now we know that ($\hbar = 1$)

$$|A,t\rangle_S = U(t,t_0)\,|A,t_0\rangle_S.$$

$$= e^{-iH(t-t_0)}\,|A,t_0\rangle_S.$$

also

$$|A,t\rangle_I = U_0^\dagger\,|A,t\rangle_S$$

$$= e^{iH_0(t-t_0)}\,e^{-iH(t-t_0)}\,|A,t_0\rangle_S$$

$$= e^{-iH_I(t-t_0)}\,|A,t_0\rangle_S.$$

But at some time long in the past or just at t_0

$$|A,t_0\rangle_I = |A,t_0\rangle_S.$$

Thus $|A,t\rangle_I = e^{-i\underset{\uparrow\ \text{hermitian}}{H_I}(t-t_0)}\,|A,t_0\rangle_S$

Thus $|\phi(t)\rangle = \overset{\text{unitary}\ \downarrow}{U}\,|\phi(t_0)\rangle$

Thus $\langle\phi(t)|\phi(t)\rangle = \langle\phi(t_0)|U^\dagger U|\phi(t_0)\rangle$

$$= \text{constant}$$

So the normalization or probability is preserved.

Introducing S- Matrix

if we define

$$|\phi(-\infty)\rangle = |i\rangle$$ ← The initial unscattered system.

The S-matrix is defined as relating the state before interaction with that after.

$$|\phi(\infty)\rangle = S|\phi(-\infty)\rangle$$

$$\equiv \quad |f\rangle = S|i\rangle$$

N.B. S is a matrix as $|i\rangle$ = a Σ of eigenstates in the initial states
$\langle f|$ = Σ of eigenstates in the final states

The transition probability that it will have started in a specific $|i\rangle$ = $\psi(-\infty)$ and ended in a specific $\langle f|$ eigenstate is

$$\propto |\langle f|\psi(+\infty)\rangle|^2$$

$\psi(+\infty) = \langle S|\psi(-\infty)\rangle$
$= S|i\rangle$

$$= |\langle f|S|i\rangle|^2 \equiv S_{fi}$$

So with the state $|\phi(\infty)\rangle$ expanded in terms of an orthonormal set of states.

$$\phi|\infty\rangle = \sum_f |f\rangle\langle f|\phi(\infty)\rangle = \sum_f |f\rangle S_{fi}.$$

clearly S is unitary

$$\langle f|f\rangle = 1$$

$$\langle i|S^{+}S|i\rangle = 1$$

since $\langle i|i\rangle = 1$ $\quad\therefore\quad S^{+}S = \mathbb{1}.$

This expresses the conservation of probability.

Calculation of S-matrix

To calculate the S-matrix we must solve

$$i \frac{d}{dt} |\phi(t)\rangle = H_I(t) \, |\phi(t)\rangle$$

[where $|\phi(t)\rangle \equiv |A, t\rangle_I$]

for the initial condition

$$|\phi(-\infty)\rangle = |i\rangle$$

Clearly the solution consistent with this condition is,

$$|\phi(t)\rangle = |i\rangle + (-i)\int_{-\infty}^{t} dt_1 \, H_I(t_1)|\phi(t_1)\rangle$$

This equation can be solved iteratively: since the term in the integral can be replaced by

$$|\phi(t_1)\rangle = |i\rangle + (-i)\int_{-\infty}^{t_1} dt_2 \, H_I(t_2)\,|\phi(t_2)\rangle$$

so

$$|\phi(t)\rangle = |i\rangle + (-i)\int_{-\infty}^{t} dt_1 \, H_I(t_1)\,|i\rangle + (-i)^2 \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_1} dt_2 \, H_I(t_1)\, H_I(t_2)|\phi(t_2)\rangle$$

At each substitution we get the original terms and a new one. This series can clear be written

$$S = \sum_{n=0}^{\infty} (-i)^n \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \cdots\cdots \int_{-\infty}^{t_{n-1}} dt_n \, H_1(t_1)\, H_1(t_2) \ldots H_1(t_n)$$

This we will find can be rewritten

$$= \sum_{n=0}^{\infty} \frac{(-i)^2}{n!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \cdots \int_{-\infty}^{\infty} dt_n \, T\left\{ H_1(t_1) H_1(t_2) \ldots H_1(t_n) \right\}$$

where $\frac{1}{n!}$ has be introduced, the integration limit extended to $+\infty$, and the time ordered product replaced the product.

To see why we can do this we will study the $n=2$ term.

$$(-i)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} H_I(t_1)\, H_I(t_2)$$

$$T\left\{ H_I(t_1)\, H_I(t_2) \right\} = \theta(t_1 - t_2)\, H_I(t_1)\, H_I(t_2) + \theta(t_2 - t_1)\, H_I(t_2) H_I(t_1)$$

$\theta(t_1 - t_2)$ Heavyside step function

So if we replace this in the term

$$= (-i)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} \Big[\theta(t_1-t_2) H_1(t_1) H_1(t_2) + \theta(t_2-t_1) H_1(t_2) H_1(t_1) \Big]$$

$$= (-i)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \, \theta(t_1-t_2) H_1(t_1) H(t_2)$$

$$+ (-i)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \, \theta(t_2-t_1) H_1(t_2) H_1(t_2)$$

Looking at the first term it can be re-written:-

$$(-i)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \underbrace{\theta(t_2-t_1) H_1(t_2) H_1(t_1)}_{=0 \text{ when } t_2 > t_1}$$

from $t_1 \to \infty$.

So we can arbitarily extend limit

The second term can have the variables interchanged

$t_1 \leftrightarrow t_2$

$$(-i)^2 \int_{-\infty}^{\infty} dt_2 \int_{-\infty}^{t_2} dt_1 \, \theta(t_1-t_2) H_1(t_1) H_1(t_2)$$

$$= (-i)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \underbrace{\theta(t_1-t_2) H_1(t_1) H_1(t_2)}_{=0 \text{ if } t_2 > t_1}$$

So for the second term substituting in $T\{H_1(t_1) H_1(t_2)\}$ gives us two terms both of which are equal to the original

Thus this term must be divided by 2.

$$\therefore (-i)^2 \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{t_1} dt_2 \quad H_1(t_1)\, H_1(t_2)$$

$$= \frac{(-i)^2}{2} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \; T\{ H_1(t_1)\, H_1(t_2)\}$$

For the third term

$$T\{ H_1(t_1)\, H_1(t_2)\, H_1(t_3)\}$$

$$\begin{aligned}
= \; & H(t_1)\, H(t_2)\, H(t_3) \quad \theta(t_1 - t_2)\ \theta(t_2 - t_3) \\
& H(t_1)\, H(t_3)\, H(t_2) \quad \theta(t_1 - t_3)\ \theta(t_3 - t_2) \\
& H(t_2)\, H(t_1)\, H(t_3) \quad \theta(t_2 - t_1)\ \theta(t_1 - t_3) \\
& H(t_3)\, H(t_1)\, H(t_2) \quad \theta(t_3 - t_1)\ \theta(t_1 - t_2) \\
& H(t_2)\, H(t_3)\, H(t_1) \quad \theta(t_2 - t_3)\ \theta(t_3 - t_1) \\
& H(t_3)\, H(t_2)\, H(t_1) \quad \theta(t_3 - t_2)\ \theta(t_2 - t_1)
\end{aligned}$$

for each term the limit of the integral can be extended because of θ functions and we can see that we can arrange the first one t_1 in 3 different ways t_2 in 2 and t_3 in 1 (ie $3 \times 2 \times 1 = 3!$)

For the nth term we get $n!$ such terms.

Thus

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \cdots \int_{-\infty}^{\infty} dt_n \, T\{H_1(t_1) H_1(t_2) \ldots H(t_n)$$

Substituting $\mathcal{H}$

$$S = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int \cdots \int d^4x_1 \ldots d^4x_n \, T\{\mathcal{H}_1(x_1) \ldots \mathcal{H}(x_n)\} .$$

since $$U(t) = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_{-\infty}^{t} dt_1 \ldots \int_{-\infty}^{t} dt_n$$

$$\mathcal{T}\left[H_1(t_1) \ldots H_1(t_n)\right]$$

since

$$S = \lim_{t \to \infty} U(t) = \mathcal{T}\left[\exp\left(-i \int_{\infty}^{\infty} dt\, H_I(t)\right)\right]$$

(compare $U(t)$ with $e^x = 1 + x + x^2/2! + \ldots$)

So $$S = \mathcal{T}\left[\exp\left(-i \int d^4x\, \mathcal{H}_I(x)\right)\right]$$

Time Ordering, Normal Ordering + Wick's Theorem

If we want to calculate the S-matrix terms we see that we need the time ordered terms.

It is very useful to express the time ordered terms as a normal ordered term + another term. It is very clear what term in the S-Matrix the normal ordered term involves.

Let us consider

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1$$

and a specific example.

$$\mathcal{H}_0 = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{\mu}\mu^2\phi^2$$

"scalar particle mass μ"

$$+ \quad \bar{\psi}(\not{\partial} + M)\psi$$

"Lepton of mass M"

$$\mathcal{H}_I = g\underbrace{\bar{\psi}i\gamma_5\psi}\phi$$

"pseudo" Scalar

Scalar.

where

$$\phi(x,t) = \int \frac{d^3k}{\sqrt{2E_k(2\pi)^3}} \left(\underbrace{e^{-ik.x} a_k}_{\phi_+} + \underbrace{e^{ik.x} a^{\dagger}}_{\phi_-} \right)$$

$$\Psi(x,t) = \int \frac{d^3p}{\sqrt{E_p(2\pi)^3}} \Big[b(p,s)\, u(p,s)\, e^{-ik.x} + d^{\dagger}(p,s)\, v(p,s)\, e^{ik.x} \Big]$$

$$\overline{\Psi}(x,t) = \int \frac{d^3p}{\sqrt{E_p(2\pi)^3}} \Big[b^{\dagger}(p,s)\, u(p,s)\, e^{-ip.x} + d(p,s)\, v(p,s)\, e^{ip.x} \Big]$$

1 Normal ordering

This requires that there are no terms where a creation operator is to the right of any annihilation operator.

So $N[a_{k_1} a^{+}_{k_2} a_{k_3} a^{+}_{k_4}] \equiv : a_{k_1} a^{+}_{k_2} a_{k_3} a^{+}_{k_4}:$

$= a^{+}_{k_2} a^{+}_{k_4} a_{k_1} a_{k_3}$.

- Now note the following

for bosons $[a_k a^{+}_k] = 1$

fermions $\{a_k a^{+}_k\} = 1$.

When writing the normally ordered form two conventions are used

i) The 1 is ignored

ii) The sign change when swapping fermions is not ignored.

$\therefore N[a_{k_1} a^{+}_{k_2}] = a^{+}_{k_2} a_{k_1}$ – bosons

$= -a^{+}_{k_2} a_{k_1}$ – fermions.

- NB. $\langle 0 | N[\ ] | 0 \rangle \equiv 0$.

The vacuum expectation value of a normal product for bosons let us just consider the product of two fields

$$\phi(x)\,\phi(y) = \begin{array}{l} \phi_+(x)\,\phi_+(y) \\ \phi_+(x)\,\phi_-(y) \\ \phi_-(x)\,\phi_+(y) \\ \phi_-(x)\,\phi_-(y) \end{array}$$

The only term here which is not normally ordered is the second where the creation operator is followed the the annihilation. To make this into the Normal ordered product we need to subtract that term and add the normally ordered.

$$\therefore\ \phi(x)\phi(y) + \overbrace{\phi_-(y)\,\phi_+(x) - \phi_+(x)\,\phi_-(y)}^{-[\phi_+(x)\,\phi_-(y)]}$$

$$= N[\phi(x)\,\phi(y)] \equiv\ :\phi(x)\,\phi(y):$$

$$\Rightarrow \boxed{\phi(x)\,\phi(y) = N[\phi(x),\,\phi(y)] + [\phi_+(x)\,\phi_-(y)]}$$

Vacuum Expectation Values of Products:

Since $\langle 0 | N[\quad] | 0 \rangle = 0$.

The physical interpretation is in creation and annihilation on 25 Prox $x \to x$.

Then

$$\overbrace{\langle 0 | \phi(x)\, \phi(y) | 0 \rangle} = \langle 0 | [\phi_+(x)\, \phi_-(y)] | 0 \rangle$$

But $[\phi_+(x)\ \phi_-(y)] \ \propto \ [a\, a^\dagger] = 1$.

↑ a number!!

thus $\langle 0 | [\phi_+(x)\, \phi_-(y)] | 0 \rangle$

$$= \underbrace{\langle 0 | 0 \rangle}_{1} [\phi_+(x), \phi_-(y)]$$

↑ a number!

∴ Therefore

$$\boxed{\langle 0 | [\phi_+(x)\, \phi_-(y)] | 0 \rangle = [\phi_+(x)\, \phi_-(y)]}$$

a number.

but also

$$\boxed{\langle 0 | \phi(x)\, \phi(y) | 0 \rangle = \langle 0 | [\phi_+(x) \phi_-(y)] | 0 \rangle}$$

∴ $\langle 0 | \phi(x)\, \phi(y) | 0 \rangle$ = a number!

THE VACUUM EXPECTATION OF PRODUCT = A NUMBER

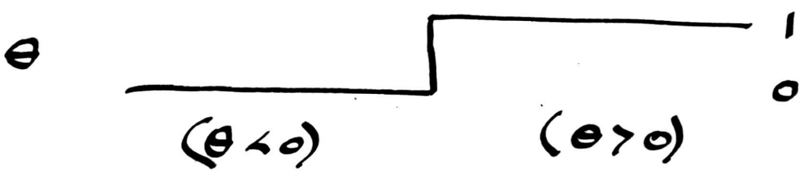
The commutator

The Time Ordered Product

By definition

$$T[\phi(x), \phi(y)] = \theta(x_0 - y_0)\,\phi(x)\,\phi(y) + \theta(y_0 - x_0)\,\phi(y)\,\phi(x)$$

where θ is the Heavyside step function



Remembering $\phi(x) = \underset{\text{annihilation}}{\phi_+(x)} + \underset{\text{creation}}{\phi_-(x)}$

and

$$\phi(x)\,\phi(y) = N[\phi(x), \phi(y)] + [\phi_+(x)\,\phi_-(y)]$$

However taking the vacuum expectation value of both sides

$$\langle 0|\,\phi(x), \phi(y)\,|0\rangle = \langle 0|[\phi_+(x)\,\phi_-(y)]\,|0\rangle$$

but because we know $\phi_+\ \phi_-$ annihilation and creation operators $[\phi_+\ \phi_-]$ is just a number!

$$= \langle 0|0\rangle\,[\phi_+(x)\,\phi_-(y)]$$

$$= [\phi_+(x)\,\phi_-(y)]$$

So putting this into the definition of the time ordered product and remembering

$$\theta(x_0 - y_0) + \theta(y_0 - x_0) = 1$$

and for bosons

$$N[\phi(x)\,\phi(y)] = N[\phi(y)\,\phi(x)]$$

* (with bosons we can swap fields around without changing the Normal ordered product)

Thus starting with

$$\phi(x)\,\phi(y) = N[\phi(x)\,\phi(y)] + \langle 0|\,\phi(x)\phi(y)\,|0\rangle$$

$$\begin{aligned} \mathcal{T}[\phi(x)\phi(y)] &= \theta(x_0 - y_0)\,\phi(x)\,\phi(y) \\ &\quad + \theta(y_0 - x_0)\,\phi(y)\,\phi(x) \\ &= \theta(x_0 - y_0)\,N[\phi(x)\,\phi(y)] + \theta(y_0 - x_0)N[\phi(y)\,\phi(x)] \\ &\quad + \langle 0|\,\theta(x_0 - y_0)\,\phi(x)\phi(y) + \theta(y_0 - x_0)\phi(y)\,\phi(x)|0\rangle \\ &= N[\phi(x)\,\phi(y)] + \langle 0|\,\mathcal{T}[\phi(x)\phi(y)]\,|0\rangle \end{aligned}$$

*

Defining

$$\boxed{\langle 0|\,\mathcal{T}[\phi(x)\,\phi(y)]\,|0\rangle \equiv \underbrace{\phi(x)\;\phi(y)}}$$

a number see below.

$$\boxed{\mathcal{T}[\phi(x), \phi(y)] = N[\phi(x)\,\phi(y)] + \underbrace{\phi(x)\,\phi(y)}}$$

* Note that

$$\langle 0|\,\phi(x)\,\phi(y)\,|0\rangle = [\phi_+(x), \phi_-(y)]$$

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The same relationship for fermions

(b) = bosons (f) = fermions

1)

(b) $T[\phi(x)\phi(y)] \equiv \theta(x_0-y_0)\phi(x)\phi(y) + \theta(y_0-x_0)\phi(y)\phi(x)$

(f) $T[\phi(x)\phi(y)] \equiv \theta(x_0-y_0)\phi(x)\phi(y) - \theta(y_0-x_0)\phi(y)\phi(x)$

for fermions fields anticommute

N.B. $\phi(x)\phi(y) = \phi_+(x)\phi_+(y) + \underbrace{\phi_+(x)\phi_-(y)} + \phi_-(x)\phi_+(y) + \phi_-(x)\phi_-(y)$

The normal operator just flips this term to

2)

(b) $N[\phi(x)\phi(y)] = \phi_+(x)\phi_+(y) + \overbrace{\phi_-(y)\phi_+(x)} + \phi_+(x)\phi_-(y) + \phi_-(x)\phi_-(y)$

(f) $N[\phi(x)\phi(y)] = \phi_+(x)\phi_+(y) \underbrace{- \phi_-(y)\phi_+(x)} + \phi_+(x)\phi_-(y) + \phi_-(x)\phi_-(y)$

again fermions anticommute.

3) So

(b) $N[\phi(x)\phi(y)] = \phi(x)\phi(y) - \phi_+(x)\phi_-(y) + \phi_-(y)\phi_+(x)$

$= \phi(x)\phi(y) - [\phi_+(x)\phi_-(y)]$

(f) $N[\phi(x)\phi(y)] = \phi(x)\phi(y) - \phi_+(x)\phi_-(y) - \phi_-(y)\phi_+(x)$

$= \phi(x)\phi(y) - \{\phi_+(x)\phi_-(y)\}$

So the relationship between the vacuum expectation value and the commutator is

$$ⓑ \quad \langle 0|\phi(x)\phi(y)|0\rangle = \langle 0|[\phi_+(x),\phi_-(y)]|0\rangle = [\phi_+(x),\phi_-(y)]$$
$$ⓕ \quad \langle 0|\phi(x)\phi(y)|0\rangle = \langle 0|\{\phi_+(x),\phi_-(y)\}|0\rangle = \{\phi_+(x),\phi_-(y)\}$$

- So in both cases the VEV is a number.

So we showed for bosons

$$T[\phi(x)\phi(y)] = N[\phi(x)\phi(y)] + \underbrace{\phi(x)\phi(y)}$$

, ie the VEV of the product.

for fermions

$$\phi(x)\phi(y) = N[\phi(x)\phi(y)] + \langle 0|\phi(x)\phi(y)|0\rangle$$
$$\phi(y)\phi(x) = -N[\phi(y)\phi(x)] - \langle 0|\phi(y)\phi(x)|0\rangle$$

putting these in for 1) ⓕ gives exactly the same equation

$$T[\phi(x)\phi(y)] = N[\phi(x)\phi(y)] + \underbrace{\phi(x)\phi(y)}$$

So in summary ① is required proof if we just use the − sign when swapping $\phi(x)\phi(y)$ for fermions.

② + ③ is the proof that $\langle 0|\phi(x)\phi(y)|0\rangle$ is a c-number for f and b

Wick's Theorem

This is the relationship between the time ordered products and the normal ordered products and for two fields it is just what we have just derived.

$$\mathcal{T}[\phi(x_1)\,\phi(x_2)] = N[\phi(x_1)\,\phi(x_2)]$$

$$+ \quad \underbrace{\phi(x_1)\,\phi(x_2)}$$

if the fields correspond to the same particle

$$\underbrace{\phi(x_1)\,\phi(x_2)} = \langle 0|\mathcal{T}[\phi(x_1)\,\phi(x_2)]|0\rangle$$

$$= i\,\Delta_F\,(x_1 - x_2)$$

$$\phi(x_1)\,\phi^{+}(x_2) = i\,\Delta_F\,(x_1 - x_2)$$

$= \phi^{+}(x_2)\,\phi(x_1)$ - for bosons.

$$\psi_\alpha(x_1)\,\overline{\psi}_\beta(x_2) = -\,\overline{\psi}_\beta(x_2)\,\psi_\alpha(x_1) \quad \text{- for fermions}$$

$$= i\,S_{F\alpha\beta}\,(x_1 - x_2)$$

where $\Delta_F(x_1 - x_2)$ and $S_{F\alpha\beta}(x_1 - x_2)$ are Feynman Propagators.

Wick's Theorem can now be generalized to any number of fields (its proof is by induction and not very enlightening so will be omitted).

$$
\begin{aligned}
& T[\phi_1(x_1) \ldots\ldots \phi_n(x_n)] \\
&= N[\phi_1 \ldots\ldots \phi_n] \\
&+ \phi_1\phi_2\, N[\phi_3 \ldots \phi_n] + \phi_1\phi_3\, N[\phi_2 \ldots \phi_n] + \ldots\ldots \\
&+ \underbrace{\phi_1\phi_2}\;\underbrace{\phi_3\phi_4}\; N[\phi_5 \ldots \phi_n] + \ldots\ldots \\
&\;\vdots \\
&+ \begin{cases} \underbrace{\phi_1\phi_2}\;\underbrace{\phi_3\phi_4} \ldots\ldots \underbrace{\phi_{n-1}\phi_n} + \ldots\ldots & \text{if } n \text{ even} \\ (\underbrace{\phi_1\phi_2}\;\underbrace{\phi_3\phi_4} \ldots\ldots \phi_{n-2}\phi_{n-1})\,\phi_n + \ldots\ldots & \text{if } n \text{ odd} \end{cases}
\end{aligned}
$$

where ..... implied "all possible permutations".

Below is the expansion for 4 fields

$$
\begin{aligned}
T\{\phi_1\phi_2\phi_3\phi_4\} &= N[\phi_1\phi_2\phi_3\phi_4] \\
&+ \underbrace{\phi_1\phi_2}_{} N[\phi_3\phi_4] + \underbrace{\phi_1\phi_3}_{} N[\phi_2\phi_4] \\
&+ \underbrace{\phi_1\phi_4}_{} N[\phi_2\phi_3] + \underbrace{\phi_2\phi_3}_{} N[\phi_1\phi_4] \\
&+ \underbrace{\phi_2\phi_4}_{} N[\phi_1\phi_3] + \underbrace{\phi_3\phi_4}_{} N[\phi_1\phi_2] \\
&+ \underbrace{\phi_1\phi_2}_{}\,\underbrace{\phi_3\phi_4}_{} + \underbrace{\phi_1\phi_3}_{}\,\underbrace{\phi_2\phi_4}_{} \\
&+ \underbrace{\phi_1\phi_4}_{}\,\underbrace{\phi_2\phi_3}_{}
\end{aligned}
$$

So the vacuum expectation value for this is

$$
\langle 0| T[\phi_1\phi_2\phi_3\phi_4] |0\rangle = \underbrace{\phi_1\phi_2}_{}\,\underbrace{\phi_3\phi_4}_{} + \underbrace{\phi_1\phi_3}_{}\,\underbrace{\phi_2\phi_4}_{} + \underbrace{\phi_1\phi_4}_{}\,\underbrace{\phi_2\phi_3}_{}
$$

So we should ask does this have a physical interpretation? First however we will examine the meaning of $\underbrace{\phi_1\phi_2}_{}$.