Transformation properties and Covariance of the Dirac equation

Non-relativistic 2-component spinors under rotation

Before we embark on looking at the full covariance of the Dirac equation under a generalized Lorentz transformation we can learn a lot about the techniques by looking at a simple non relativistic 2-component spinor under rotation. Non-relativistic equations still have to be invariant under rotation. A simple hamiltonian that expresses the energy of a spin ½ particle in a B field can be written as

$$\sigma . B \chi = E \chi$$

where χ is a 2-component spinor.

If the term $\chi^{+}\chi$ is to be invariant under rotation

$$\chi^{+}\chi = \chi^{+\prime}\chi^{\prime}$$

where $\chi' =$ spinor χ in rotated frame.

if $\chi = U\chi'$ where U is some transformation matrix

$$\chi^{+} = \widetilde{U\chi'}^{*} = \chi'^{+}U^{+}$$

Therefore $\chi^{+\prime}\chi' = \chi^{+}U^{+}U\chi = \chi^{+}\chi$

if $\chi^{+}\chi$ is to be invariant $U^{+}U = \mathbb{1}$.

Thus $\boxed{U^{+} = U^{-1}}$ A matrix with this property is called a unitary transformation

N.B. $U^{+} = U$ is a hermitian matrix, so a matrix which is hermitian and unitary is self inverse $U^{+} = U = U^{-1}$.

The exponent of a matrix.

e^A may seem a strange concept but remembering that

$$e^x = 1 + x + \frac{x^2}{2!} + \dots \frac{x^n}{n!} \dots$$

where x is a simple scalar, we can express the exponent of the matrix in the same way.

$$e^A = \mathbb{1} + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \dots\dots$$

So in the same way as for the scalar the exponent of a matrix is a sum of matrixes.

The link between hermitian and unitary matrices.

let $U = e^A$ $\quad U^+ = (e^A)^+ = e^{A^+}$

(think of the effect of $\tilde{U}^* = U^+$ on the matrix sum above :-

$$e^{\tilde{A}^*} = \tilde{\mathbb{1}}^* + \tilde{A}^* + \frac{1}{2!} \tilde{A}^* \dots$$
$$= \mathbb{1} + A^+ + \frac{1}{2!} A^+ \qquad = e^{A^+}.$$

Thus if U were to be unitary

$$U^{+}U = \mathbb{1} = e^{A}.e^{A^{+}} = e^{A+A^{+}}.$$

thus A^{+} must be $= -A$. (anti hermitian)

if we express A as iA instead we get.

$$U^{+}U = \mathbb{1} = e^{iA}e^{-iA^{+}} = e^{i(A-A^{+})}$$

Thus $A = A^{+}$ = hermitian.

Thus we can build a unitary matrix transformation from a hermitian matrix.

$$\boxed{U = e^{iA}}$$

$A =$ hermitian
$A^{+} = A$.

But we know that $\sigma_1\ \sigma_2\ \sigma_3\ \mathbb{1}$ form a complete set of 2×2 hermitian matrices.

Let us try

$$U = \exp(i\sigma_3\,\theta/2)$$
$$U^+ = \exp(-i\sigma_3\,\theta/2)$$

we might guess that this transformation is connected to a rotation about the z-axis as σ_3 is associated with σ_z. But why the $\theta/2$ ... we shall see.

<u>Invariance under rotation of the 2-component spinor energy equation</u>

1) $\sigma.B\chi = E\chi$

under rotation

2) $\sigma.B'\chi' = E\chi'$ where $\chi' = U\chi$

multiplying 1) by U.

$$U\sigma.B\chi = EU\chi$$

and inserting $U^+U = \mathbb{1}$

$$U\sigma.BU^+U\chi = EU\chi.$$

But $\sigma . B$ just represents a sum of 3 terms $\sigma_x B_x + \sigma_y B_y + \sigma_z B_z$ and the B component is just a scalar in each. Thus the equation can be written as.

$$U \sigma U^{+} . B \chi' = E \chi'$$

Thus we can see that if this equation is to be invariant under rotation

$$\boxed{U \underline{\sigma} U^{+} B = \sigma . B'} \qquad \therefore \quad U \sigma U^{+} B = R(\theta) \underline{B} . \underline{\sigma}$$

But remembering that B under rotation about the z-axis transforms as.

$$B' = \begin{pmatrix} B_x' \\ B_y' \\ B_z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}$$

Let us see what

$$U = \exp(i \sigma_z \theta/2)$$ gives

$$U = \exp\left(i\sigma_z \theta/2\right)$$

$$= \mathbb{1} + \left(i\sigma_z \frac{\theta}{2}\right) + \frac{1}{2!}\left(i\sigma_z \frac{\theta}{2}\right)^2 + \frac{1}{3!}\left(i\sigma_z \frac{\theta}{2}\right)^3 + \cdots$$

The even terms give me

$$\mathbb{1}\left(\cos\theta/2\right) \qquad \text{since } \sigma_3^{\,2} = \mathbb{I} .$$

The odd terms give

$$i\sigma_3\left(\sin\theta/2\right)$$

$$\therefore U = \mathbb{1}\cos\theta/2 + i\sigma_3 \sin\theta/2$$

$$= \begin{pmatrix} \cos\theta/2 & 0 \\ 0 & \cos\theta/2 \end{pmatrix} + \begin{pmatrix} i\sin\theta/2 & 0 \\ 0 & -i\sin\theta/2 \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix}$$

$U\ \sigma.B\ U^{+}$

$$= \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \begin{pmatrix} B_z & B_x - iB_y \\ B_x + iB_y & -B_z \end{pmatrix} \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\theta/2} & 0 \\ 0 & e^{-i\theta/2} \end{pmatrix} \begin{pmatrix} B_z e^{-i\theta/2} & (B_x - iB_y)e^{i\theta/2} \\ (B_x + iB_y)e^{-i\theta/2} & -B_z e^{i\theta/2} \end{pmatrix}$$

$$= \begin{pmatrix} B_z & (B_x - iB_y)e^{i\theta} \\ (B_x + iB_y)e^{-i\theta} & -B_z \end{pmatrix}$$

$$= \begin{pmatrix} B_z & (B_x\cos\theta + B_y\sin\theta) - i(B_y\cos\theta - B_x\sin\theta) \\ (B_x\cos\theta + B_y\sin\theta) + i(B_y\cos\theta - B_x\sin\theta) & -B_z \end{pmatrix}$$

Since $B_x' = B_x\cos\theta + B_y\sin\theta \quad B_y' = B_y\cos\theta - B_x\sin\theta$

$B_z' = B_z$

$$= \begin{pmatrix} B_z' & B_x' - iB_y' \\ B_x' + iB_y' & -B_z' \end{pmatrix}.$$

which is what is required for invariance.

We can see that the spinor is a strange object as.

$$\chi' = U\chi = \exp[i\,\sigma_3\,\theta/2]\,\chi .$$

Thus a spinor rotated through 360° is not the same as one that has not been rotated. To get this back to its starting state we have to rotated by 2×360°.

Does this matter?

It is possible to show that for an arbitrary rotation about $\underline{n}$ the spinor transformation is

$$\boxed{U = \exp[i\,\sigma.n\,\theta/2]}$$

The Dirac Equation and 4-spinors

Lorentz covariance

Rotation is a special case of the general Lorentz transformation from the Poincaré Group.

So what happens to the Dirac Equation

$$\hat{E}\psi(x,t) = (\alpha.p + \beta m)\psi(x,t) \quad -(1)$$

when it is subject to a transformation

$$\psi'(x',t) = S_R\psi(x,t)$$

↳ R means a rotational sub transformation.

$$\hat{E}\psi'(x',t) = (\alpha.p' + \beta m)\psi'(x',t) \quad -(2)$$

So substituting S_R, or rather operation by S_R on both sides of (1) we get

$$\hat{E}S_R\psi(x,t) = S_R(\alpha.p + \beta m)\psi(x,t)$$

$S_R^{-1}S_R = \mathbb{1}$.

$$\boxed{\hat{E}S_R\psi(x,t) = S_R(\alpha.p + \beta m)S_R^{-1}S_R\psi(x,t)}$$

$$\therefore \hat{E}\, S_R \Psi(x,t) = \left(S_R \underline{\alpha} \cdot S_R^{-1} \cdot \underline{P} + S_R \beta S_R^{-1} \right) S_R \Psi(x,t)$$

Thus if equation 2 is to be the same as 1.

$$\boxed{\begin{aligned} (S_R \alpha S_R^{-1}) \cdot P &= \underline{\alpha} \cdot P' \\ S_R \beta S_R^{-1} &= \beta \end{aligned}}$$

$\Leftarrow$ This is the defining equation for Lorentz invariance.

Thus the equations are the same as for the 2-component spinor so it is no surprise that it can be expanded to the 4 component case by writing

$$S_R = \begin{pmatrix} \exp\left(i + \sigma \cdot n\, \theta/2\right) & 0 \\ 0 & \exp\left(+i \sigma \cdot n\, \theta/2\right) \end{pmatrix}$$

clearly again $\quad S_R^{+} = S_R^{-1}$

$$\therefore S_R^{+} S_R = \mathbb{1} .$$

Dirac Equation and the Lorentz Transformation

If we define a generalized Lorentz transform Λ

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

4-vectors.

$$x'_{\mu} = \Lambda_{\mu}{}^{\nu} x_{\nu}$$

But $x'^{\mu} = g^{\mu\beta} x'_{\beta} = \Lambda^{\mu}{}_{\nu} g^{\nu\alpha} x_{\alpha}.$

But $g_{\delta\mu} g^{\mu\beta} = g_{\delta}{}^{\beta} = \delta_{\delta}^{\beta} = 1.$

$$\therefore \quad g_{\delta\mu} g^{\mu\beta} x'_{\beta} = g_{\delta\mu} \Lambda^{\mu}{}_{\nu} g^{\nu\alpha} x_{\alpha}.$$

$$\delta_{\delta}^{\beta} x'_{\beta} = g_{\delta\mu} \Lambda^{\mu}{}_{\nu} g^{\nu\alpha} x_{\alpha}.$$

$$\therefore \quad x'_{\delta} = \underbrace{g_{\delta\mu} \Lambda^{\mu}{}_{\nu} g^{\nu\alpha}}_{\Lambda_{\delta}{}^{\alpha}} x_{\alpha}.$$

$$\boxed{\Lambda_{\delta}{}^{\alpha} = g_{\delta\mu} \Lambda^{\mu}{}_{\nu} g^{\nu\alpha}}$$

Consider the inner product.

$$A.B = A'.B'$$

$$A'^{\mu} = \Lambda^{\mu}{}_{\alpha} A^{\alpha} \qquad B'^{\nu} = \Lambda^{\nu}{}_{\beta} B^{\beta}$$

$$A'.B' = A'_{\nu} B^{\nu'} = g_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} A^{\alpha} B^{\beta}$$

$$A.B = \underbrace{A_{\beta}}_{g_{\alpha\beta} A^{\alpha}} B^{\beta} = g_{\alpha\beta} A^{\alpha} B^{\beta}.$$

Thus $g_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} A^{\alpha} B^{\beta} = g_{\alpha\beta} A^{\alpha} B^{\beta}.$

$$\therefore \boxed{g_{\mu\nu} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} = g_{\alpha\beta}.}$$

But $g_{\mu\nu} \Lambda^{\mu}{}_{\alpha} = \Lambda_{\nu\alpha}$

since $A_{\nu} = g_{\mu\nu} A^{\mu}$, $A^{\nu} = g^{\nu\mu} A_{\mu}$. Are also true as $g^{\nu}{}_{\mu}$ $g_{\nu\mu}$ are symmetric and diagonal.

$$\therefore \Lambda_{\nu\alpha} \Lambda^{\nu}{}_{\beta} = g_{\alpha\beta}.$$

$$\therefore g^{\rho\alpha} \Lambda_{\nu\alpha} \Lambda^{\nu}{}_{\beta} = g^{\rho\alpha} g_{\alpha\beta}.$$

$$\Rightarrow \boxed{\Lambda_{\nu}{}^{\rho} \Lambda^{\nu}{}_{\beta} = g^{\rho}{}_{\beta} = \delta_{\rho\beta}}$$

Thus, since

$$\Lambda_{\gamma}{}^{\rho} \Lambda^{\gamma}{}_{\beta} = g^{\rho}{}_{\beta} = \delta_{\rho\beta}$$

$$(\Lambda^{-1})^{\rho}{}_{\gamma} \Lambda^{\gamma}{}_{\beta} = \delta_{\rho\beta} .$$

$$\therefore \; (\Lambda^{-1})^{\rho}{}_{\gamma} = \Lambda_{\gamma}{}^{\rho}$$

Thus if we know $\Lambda_{\gamma}{}^{\rho}$ we can find Λ^{-1}!
To get Λ^{-1} from $\Lambda_{\gamma}{}^{\rho}$ we need to convert it to $\Lambda^{\gamma}{}_{\rho}$ and then transpose it $\Lambda^{\rho}{}_{\gamma}$.

$$\therefore \; \Lambda^{-1} = \left(g^{\delta\gamma} \Lambda_{\gamma}{}^{\rho} g_{\rho\sigma} \right)^{T} .$$

You can see from this that since $\det g = +1$ the determinant of Λ is ± 1.

+1 is a proper Lorentz transformation

−1 is an improper or discrete transformation.

Thus considering the Dirac Equation

$$(\not{p} - m)\psi = 0.$$

$$(\gamma^\mu p_\mu - m)\psi = 0.$$

Under transformation

$$(\gamma^\mu p_\mu' - m)\psi'(x') = 0.$$

where $p_\mu' = \Lambda_\mu{}^\nu p_\nu$; $\psi'(x') = S\psi(x)$.

Thus starting with

$$(\gamma^\mu p_\mu - m\mathbb{1})\psi = 0.$$

$$S(\gamma^\mu p_\mu - m\mathbb{1})\psi = 0.$$

$$S(\gamma^\mu p_\mu - m\mathbb{1})S^{-1}S\psi = 0.$$

$$= (\underbrace{S\gamma^\mu S^{-1} p_\mu}_{\gamma^\mu \Lambda_\mu{}^\nu p_\nu} - m\mathbb{1})\psi'(x') = 0.$$

Thus $\boxed{S\gamma^\mu S^{-1} = \Lambda^\mu{}_\nu \gamma^\nu}$

So the equation becomes.

$$\boxed{\gamma^\nu \Lambda^\mu{}_\nu p_\mu = \gamma^\nu p_\nu'}$$

The proper Lorentz transformations which can be made up from small increments (ie not discrete) can either be rotations or boosts. We have looked at rotations but now let us consider boosts. The Lorentz boost in the x direction is given by.

$$\begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} = \begin{pmatrix} \cosh w & \sinh w & 0 & 0 \\ \sinh w & \cosh w & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\tan w = \beta$

we can immediately see that S_B is analagous to S_R but

$\sigma \to \alpha$. matrices

$\theta \to iw$ complex angle

we find that

$$S_B = e^{\alpha_1 w/2}$$

is a boost along the x direction

$$e^{\alpha' w/2} = \cosh(w/2)\, \mathbb{1} + \sinh(w/2)\, \alpha'$$

for an arbitrary boost in direction v.

$$S_B = e^{\left\{ \frac{\alpha \cdot v}{|v|} (w/2) \right\}}$$

A scalar current probability.

From the definition of the Dirac probability

$\bar{\psi}(x)\psi(x)$ must be a scalar probability.

if $\psi'(x') = S\psi(x)$.

$$\overline{\psi'(x')}\,\psi'(x) = \left(S\psi(x)\right)^{+}\gamma^{0}S\psi(x)$$

$$= \psi^{+}(x)\,S^{+}\gamma^{0}S\psi(x).$$

$$= \psi^{+}(x)\gamma^{0}\gamma^{0}S^{+}\gamma^{0}S\psi(x).$$

$$= \bar{\psi}(x)\,\gamma^{0}S^{+}\gamma^{0}S\psi(x).$$

Now as $\gamma^{0}S^{+}\gamma^{0} = S^{-1}$

$$= \bar{\psi}(x)\,S^{-1}S\psi(x).$$

$$= \bar{\psi}(x)\psi(x)$$

$\therefore$ scalar.

Thus

N.B

$$\boxed{\begin{aligned} \gamma^{0}S^{+}\gamma^{0} &= S^{-1} \\ S^{-1}\gamma^{\mu}S &= A^{\mu}{}_{\nu}\gamma^{\nu} \end{aligned}}$$

Covariance of the probability current $j^\mu = \bar{\psi}\gamma^\mu\psi$

$$
\begin{aligned}
j^{\mu\prime}(x') &= \bar{\psi}'(x')\gamma^\mu\psi'(x') \\
&= (S\psi(x))^\dagger\gamma^0\gamma^\mu S\psi(x) \\
&= \psi^\dagger(x)\, S^\dagger\gamma^0\gamma^\mu S\psi(x) \\
&= \psi^\dagger(x)\,\gamma^0 S^{-1}\gamma^0\gamma^0\gamma^\mu S\psi(x). \\
&= \psi^\dagger(x)\gamma^0 S^{-1}\gamma^\mu S\psi(x). \\
&= \bar{\psi}(x)\, S^{-1}\gamma^\mu S\psi(x). \\
&= \Lambda^\mu{}_\nu\,\bar{\psi}(x)\gamma^\nu\psi(x).
\end{aligned}
$$

Thus the terms of the $j^\mu(x)$ are truely a 4-vector.

$$\gamma^5 = i\,\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{1}{4!}\,\varepsilon_{abcd}\,\gamma^a\gamma^b\gamma^c\gamma^d$$

using the property $\quad S^{-1}\gamma^a S = \Lambda^a{}_b\gamma^b$.

We can expand

$$S^{-1}\gamma^a\gamma^b\gamma^c\gamma^d S$$

$$= S^{-1}\gamma^a S\,S^{-1}\gamma^b S\,S^{-1}\gamma^c S\,S^{-1}\gamma^d S$$

$$= \Lambda^a{}_e\,\Lambda^b{}_f\,\Lambda^c{}_g\,\Lambda^d{}_h\,\gamma^e\gamma^f\gamma^g\gamma^h$$

Thus $\frac{1}{i}\,S^{-1}\gamma^5 S = \frac{1}{4!}\,\varepsilon_{abcd}\,\Lambda^a{}_e\Lambda^b{}_f\Lambda^c{}_g\Lambda^d{}_h\,\gamma^e\gamma^f\gamma^g\gamma^h$

But considering the properties of the γ's and Λ's.

$$\{\gamma^e\gamma^f\} = 2g^{ef}$$

But we also know

$$\Lambda^a{}_e\Lambda^b{}_f\,g^{ef} = g^{ab}$$ — see page 13.

Thus combining this

$$\Lambda^a{}_e\Lambda^b{}_f\{\gamma^e\gamma^f\} = 2g^{ab}$$

$$\therefore\ \varepsilon_{abcd}\,\Lambda^a{}_e\Lambda^b{}_f\{\gamma^e\gamma^f\} = 2\,\varepsilon_{abcd}\,g^{ab} = 0.$$

Thus the complete expression

$$\frac{1}{4!} \varepsilon_{abcd} \Lambda^a{}_e \Lambda^b{}_f \Lambda^c{}_g \Lambda^d{}_h \gamma^e \gamma^f \gamma^g \gamma^h$$

is completely antisymmetric under the interchange of any of e f g h.

(note that if any of e f g h are repeated that would not be true so e f g h must all be different)

So we can introduce ε_{efgh} to make this true and then we only need one example of $\gamma^e \gamma^f \gamma^g \gamma^h$ which will be the same for all

$$\therefore \varepsilon_{efgh} \gamma^e \gamma^f \gamma^g \gamma^h = \varepsilon_{efgh} \gamma^0 \gamma^1 \gamma^2 \gamma^3$$

Thus

$$\frac{1}{i} S^{-1} \gamma^5 S = \frac{1}{4!} \varepsilon_{abcd} \Lambda^a{}_e \Lambda^b{}_f \Lambda^c{}_g \Lambda^d{}_h \varepsilon^{efgh} \frac{\gamma^5}{i}$$

$$\therefore \boxed{S^{-1} \gamma^5 S = \det \Lambda \, \gamma^5}$$

This is a pseudo scalar.

Discrete Symmetries.

Parity.

$$\hat{E}\,\psi(\underline{x},t) = (\alpha.p + \beta m)\,\psi(\underline{x},t)$$

$$\underline{x} \to \underline{x}' = -\underline{x}\ ;\ t \to t' = t$$

we need to show.

$$\hat{E}\,\psi_p(\underline{x}',t') = (\underline{\alpha}.\underline{p}' + \beta m)\,\psi_p(x',t')$$

where ψ_p is the new wave function

Now $\hat{p}' = -i\nabla' = +i\nabla = -\hat{p}$

since $x' = -x$
$t' = t$

So the second equation can be written

$$\hat{E}\,\psi_p(\underline{x}',t) = (-\alpha.\hat{p} + \beta m)\,\psi_p(x',t)$$

However if we multiply the equation through by β.

$$\hat{E}\,\beta\psi_p(x',t) = \beta(-\alpha.p + \beta m)\,\psi_p(x',t).$$

But $-\beta\alpha = \alpha\beta$ $\quad (\alpha\beta + \beta\alpha) = 0$ anticommute

Thus

$$\hat{E}\,\beta\psi_p(x',t) = (\alpha.p + \beta m)\,\beta\psi_p(x',t).$$

$$\therefore \boxed{\psi_p(x',t) = \beta\psi(x,t).}$$

Time Reversal.

$$i \frac{\partial \psi}{\partial t}(\underline{x}, t) = \left(i \underline{\alpha} . \underline{\nabla} + \beta m\right) \psi(x, t).$$

$$\psi_T(x') \equiv \psi_T(x, -t).$$

need to find.

$$i \frac{\partial \psi_T}{\partial t'}(x') = \left(-i \alpha . \nabla + \beta m\right) \psi_T(x')$$

$$\frac{\partial}{\partial t'} = -\frac{\partial}{\partial t}$$

$$-i \frac{\partial \psi_T}{\partial t}(x') = \left(-i \alpha . \nabla + \beta m\right) \psi_T(x')$$

Taking the complex conjugate of the top equation:

$$-i \frac{\partial \psi^*}{\partial t}(x, t) = (i \alpha^* . \nabla + \beta^* m) \psi^*(x, t)$$

Charge Conjugation

$$i \frac{\partial \psi}{\partial t} = \left(-i \alpha . \nabla + e \alpha . A - e A^0 + \beta m \right) \psi$$

$$i \frac{\partial \psi_c}{\partial t} = \left(-i \alpha . \nabla - e \underline{\alpha} . \underline{A} + e A^0 + \beta m \right) \psi_c .$$

Take complex conjugate and multiplying by $\gamma^2 \equiv \beta \alpha_2$.

we find

$$\psi_c = i \beta \alpha_2 \psi^* (x) = i \gamma^2 \psi^* (x) .$$