## How to include the electromagnetic interaction in the Dirac equation:

We can include the electromagnetic 4-vector in the free space Dirac equation via the "minimal prescription" as for the K.G. equation.

$$\hat{p}^\mu \longrightarrow \hat{p}^\mu + eA^\mu \equiv (\partial^\mu + eA^\mu)$$

where $A^\mu = (\phi, \underline{A})$
electromagnetic vector pot.
electrostatic potential.

Thus

$$i\frac{\partial}{\partial t}\psi = (-i\alpha.\nabla - \beta m)\psi$$

becomes

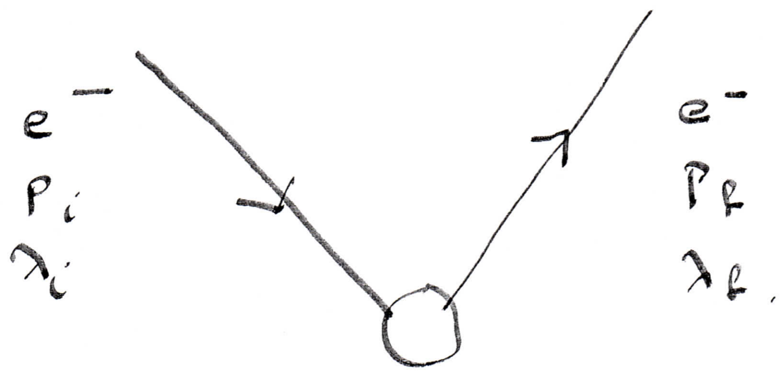
$$-i\left(\frac{\partial}{\partial t} + e\phi\right)\psi = i(\alpha.(\nabla + e\underline{A}) - \beta m)\psi$$

$$\therefore -i\frac{\partial\psi}{\partial t} = i(\alpha.\nabla - \beta m)\psi + i(e\underline{\alpha}.\underline{A} - e\phi)\psi$$

Thus we can see the electromagnetic potential is $(e\underline{\alpha}.\underline{A} - e\phi)\psi$

which can be used as a perturbation potential if it is small enough, note it operates on the spinors.

Thus we can write down the amplitude of $e^-$ coulom scattering.




$$a_f^{(1)} = i \int dt \int d^3x \, \psi_f^{+}(e^-, p_f, \lambda_f) \, V^{(1)} \psi(e^- p_f \lambda_f).$$

$$\text{where} \quad V^{(1)} = e(\alpha . A - A^0 \mathbb{1}).$$

So known the normalization we must use for the Dirac wave function we can write

$$a_f^{(1)} = -i \int \frac{e^{i p_f^{\mu} x_{\mu}}}{L^{3/2}} \left(\frac{M}{E_f}\right)^{1/2} u_f(p_f, \lambda)$$

$$\cdot \, e(\alpha . A - \mathbb{1} A^0)$$

$$\left(\frac{M}{E_i}\right)^{1/2} \frac{e^{-i p_i^{\mu} x_{\mu}}}{L^{3/2}} u_i(p_i, \lambda_i) \, d^3x \, dt.$$

In the case where the interaction is a simple static coulomb interaction

$$A^0 = \phi = \frac{eZ}{4\pi|x|} \qquad \underline{A} = 0 .$$

$$\therefore \quad a_{if}^{(1)} = -i2\pi\,\delta(E_f - E_i)\,V_{if} \qquad q = (p_i - q_i)$$

$$\text{where } V_{if} = \frac{m}{EL^3}\underbrace{\left\{\int e^{-iq.x}\,\frac{-Ze^2}{4\pi|x|}\,d^3x\right\}}_{\text{fourier transform.}}\,u_f^{+}u_i$$

$$V_{if} = \frac{Ze^2}{L^3 q^2}\,\frac{m}{E}\,u_f^{+}u_i$$

but the transition rate is $|a_{if}^{(1)}|^2$

$$R_{if} = 2\pi\,N_f(E)\,|V_{fi}|^2 \qquad (E_i = E_f).$$

$$\text{or } d\sigma = \frac{2\pi\,N_f(E)\,|V_{if}|^2\,E_i}{|p_i|}$$

$$\frac{d\sigma}{d\Omega_f} = \frac{4Z^2e^4m^2}{q^4}\,|u_f^{+}u_i|^2$$

So the rate / cross section is proportional to

$|u_f^{+} u_i|^2$.

But in a typical unpolarized experiment we add up all the final states and it is an average over the initial polarization states.

1) Average over initial polarization states
2) Sum over the final polarization states.

$$S = \frac{1}{2} \sum_{\lambda_i} \sum_{\lambda_f} |u^{+}(p_f, \lambda_f)\, u(p_i, \lambda_i)|^2$$

N.B. $u = \begin{pmatrix} \phi \\ \frac{\sigma . p}{E+M} \phi \end{pmatrix}$ $\phi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\phi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$= \frac{1}{2} \{ |u^{+}(p_f, \lambda_1)\, u(p_i, \lambda_1)|^2$$
$$+ |u^{+}(p_f, \lambda_2)\, u(p_i, \lambda_1)|^2$$
$$+ |u^{+}(p_f, \lambda_1)\, u(p_i, \lambda_2)|^2$$
$$+ |u^{+}(p_f, \lambda_2)\, u(p_i, \lambda_2)|^2$$

So looking at one term we get

$$u(p_f, \lambda_1) = \left(\frac{E+m}{2m}\right)^{1/2} \begin{pmatrix} \phi' \\ \frac{\sigma . p}{E+m}\phi' \end{pmatrix}$$

$\therefore$ $u^\dagger(p_f, \lambda_1)\, u(p_i, \lambda_1)$

$$= \frac{E+m}{2m} \phi'^\dagger \left\{ \mathbb{1} + \frac{(\sigma . p_f)(\sigma . p_i)}{(E+m)^2} \right\} \phi'$$

but since $(\underline{\sigma} . \underline{a})(\underline{\sigma} . \underline{b}) = \underline{a}.\underline{b}\, \mathbb{1} + i\underline{\sigma}.(\underline{a} \times \underline{b})$

$$= \frac{E+m}{2m} \phi'^\dagger \left\{ \mathbb{1} + \frac{\underline{p_i}.\underline{p_f}\, \mathbb{1}}{(E+m)^2} + \frac{i\, \underline{\sigma}.(\underline{p_f} \times \underline{p_i})}{(E+m)^2} \right\} \phi'$$

But this is horrible!

We need to square this term and add three more. Whereas we can proceed this way there is a better way!

## Introducing $\gamma$ matrices into the Dirac Equation 

The Dirac Equation can be expressed as

$$i\frac{\partial}{\partial t}\psi = (-i\alpha.\nabla + \beta m)\psi.$$

but this is slightly asymmetric in the $\left(\frac{\partial}{\partial t}, \underline{\nabla}\right) \equiv \partial^\mu$ operator. $\underline{\nabla}$ is multiplied by $\alpha$ by $\frac{\partial}{\partial t}$ is not multiplied by anything.

Premultiply the equation by $\beta$ we get.

$$i\beta\frac{\partial}{\partial t}\psi = (-i\beta\underline{\alpha}.\underline{\nabla} + \beta^2 m)\psi$$

$$= i\beta\frac{\partial}{\partial t}\psi = (-i\beta\underline{\alpha}.\underline{\nabla} + m\mathbb{1})\psi. \qquad \beta^2 = \mathbb{1}.$$

Let $\gamma^0 = \beta$

$\gamma^i = \beta\alpha^i$

we can rewrite the above as.

$$i\gamma^0\frac{\partial}{\partial t}\psi = (-i\underline{\gamma}.\underline{\nabla} + m\mathbb{1})\psi.$$

$$i\left(\gamma^0 \frac{\partial}{\partial t} + \underline{\gamma}\cdot\underline{\nabla}\right)\psi - m\mathbb{1}\psi = 0.$$

$$= \quad i(\gamma^\mu \partial_\mu - m\mathbb{1})\psi = 0$$

where $\gamma^\mu = (\gamma^0, \underline{\gamma})$ $\quad \partial^\mu = \left(\frac{\partial}{\partial t}, -\underline{\nabla}\right)$

if we introduce the slash notation for A

$$\boxed{\not{A} = \gamma^0 A^0 - \underline{\gamma}\cdot\underline{A}}$$

Thus we can see that the Dirac Equation can be rewritten in a very simple form.

$$(\not{p} - m\mathbb{1})\psi = 0.$$

<u>for the +E solution $\psi^{1,2} = u^{1,2} e^{-ip^\mu x_\mu}$</u>

satisfies $(\not{p} - m)u^{1,2} = 0.$

<u>for the -E solution $\psi^{3,4} = u^{3,4} e^{+ip^\mu x_\mu}$.</u>

satisfies $(\not{p} + m)u^{3,4} = 0.$

$u^{3,4}$ is usually rewritten as v.

$$\therefore (\not{p} + m)v = 0.$$

## Properties of the $\gamma$ matrices



we have defined

$$\gamma^0 = \beta \qquad \gamma^i = \beta\alpha^i$$

and we know that

$$\alpha^+ = \alpha \;, \quad \beta^+ = \beta \;. \quad \alpha^2 = \mathbb{1} \;, \quad \beta^2 = \mathbb{1} \;.$$

and $\alpha^i\alpha^j + \alpha^j\alpha^i = 0 \;, \quad \beta\alpha^i + \alpha^i\beta = 0 \;.$

1) for $\nu \neq 0, \mu \neq 0$.

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu$$

$$= \beta\alpha^\mu\beta\alpha^\nu + \beta\alpha^\nu\beta\alpha^\mu$$

$$= -\alpha^\mu\beta\beta\alpha^\nu - \alpha^\nu\beta\beta\alpha^\mu$$

$$= -\alpha^\mu\alpha^\nu - \alpha^\nu\alpha^\mu = -2\delta_{ij}\mathbb{1} \;.$$

2) for $\nu = \mu = 0$.

$$\beta^2 + \beta^2 = 2\mathbb{1}$$

any cross terms between $\nu = \mu = 0$ and $\nu \neq 0 \mu \neq 0$.
will also be zero.

$$\therefore \boxed{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\nu\mu}\mathbb{1}}$$

Are the $\gamma$'s hermitian?

$$\gamma^{i\dagger} = (\beta\alpha^i)^\dagger = \alpha^i\beta = -\beta\alpha^i = -\gamma \quad i \neq 0.$$

thus $\boxed{i \neq 0 \quad \gamma^{i\dagger} = -\gamma^i \quad \text{(anti hermitian)}}$

For $i \neq 0$ $\gamma^{0\dagger} = \gamma^0$ (hermitian)

This is slightly inconvenient to use as we do not want to write the maths dependent on the value of $i$. However the operation of taking the hermitian has the effect of moving $\beta$ from the front to the rear of $\alpha$. The same could be achieved by pre and post multiplication of $\beta$.

$$\therefore \gamma^\dagger = (\beta\alpha)^\dagger = \alpha\beta.$$

is identical to

$$\beta\gamma\beta = \beta\beta\alpha\beta = \alpha\beta. \quad \text{if } i \neq 0.$$

clearly it also trivially is true if $i = 0$.

$$\therefore \boxed{\gamma^\dagger = \beta\gamma\beta}$$

$$\therefore \boxed{\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0}$$

## normalization

$$\gamma^{0\,2} = \beta^2 = \mathbb{1}.$$

$$\gamma^{\mu\,2} = \beta\alpha^\mu\beta\alpha^\mu = -\alpha^\mu\beta\beta\alpha^\mu = -\mathbb{1}.$$

## the bar notation

$$\bar{\Psi} = \Psi^+\gamma^0$$

$$\bar{u} = u^+\gamma^0.$$

Having introduced the quantity

$$S = \frac{1}{2}\sum_{if} | u^{+}(p_f,\lambda_f)\, u\,(p_i,\lambda_i)|^2$$

we can see what difference the introduction of the $\gamma$ matrices has made to the calculation.

$$\therefore\ S = \frac{1}{2}\sum_{f} \left( (u\,(p_f,\lambda_f)\, u\,(p_i,\lambda_i)\right)^{+} \left(u^{+}(p_f,\lambda_f)\,u(p_i\,\lambda_i)\right)$$

$$= \frac{1}{2}\sum_{if} u^{+}(p_f,\lambda_f)\, u(p_i,\lambda_i)\, u^{+}(p_i,\lambda_i)\, u\,(p_f,\lambda_f)$$

$$= \frac{1}{2}\sum_{if} \bar{u}\,(p_f,\lambda_f)\,\gamma^{0}\, u\,(p_i,\lambda_i)\,\overline{u}^{+}(p_i,\lambda_i)\gamma^{0} u\,(p_f,\lambda_f)$$

what form does this equation take

$$(\ \bar{u}\ )\begin{pmatrix} & & \\ & \gamma^{0} & \\ & & \end{pmatrix}\begin{pmatrix} \\ u \\ \\ \end{pmatrix}(\ \bar{u}\ )\begin{pmatrix} & & \\ & \gamma^{0} & \\ & & \end{pmatrix}\begin{pmatrix} \\ u \\ \\ \end{pmatrix}$$

but rewriting this in an explicit matrix form.

$$S = \frac{1}{2}\sum_{if} \overline{u}_{\alpha}\,(p_f\,\lambda_f)\,\gamma^{0}_{\alpha\beta}\, u_{\beta}(p_i,\lambda_i)\,\overline{u}_{\gamma}(p_i,\lambda_i)\gamma^{0}_{\gamma\delta}\,\overline{u}_{\delta}(p_f\,\lambda_f)$$

But notice the central term contains all the dependence on $i$.

$$S = \frac{1}{2}\sum_{if} \bar{u}_\alpha(p_f,\lambda_f)\,\gamma^0_{\alpha\beta}\underbrace{u(p_i,\lambda_i)_\beta\;\bar{u}_\gamma(p_i,\lambda_i)}\,\gamma^0_{\gamma\delta}\,u_\delta(p_f,\lambda_f)$$

$\therefore$ Calculating just this term.

$$\sum_{i=1,2} u_\beta(p_i\lambda_i)\,\bar{u}_\gamma(p_i\lambda_i)$$

$$= u_\beta(p_i,\phi^1)\,\bar{u}_\gamma(p_i,\phi^1) + u_\beta(p_i,\phi^2)\,\bar{u}_\gamma(p_i\,\phi^2)$$

$$= \frac{E+M}{2M}\left\{\begin{pmatrix}\phi^1\\ \frac{\sigma.p}{E+M}\phi^1\end{pmatrix}\left(\phi^{1+},\ -\phi^{1+}\frac{\sigma.p}{E+M}\right) + \begin{pmatrix}\phi^2\\ \frac{\sigma.p}{E+M}\phi^2\end{pmatrix}\left(\phi^{2+},-\phi^{2+}\frac{\sigma.p}{E+M}\right)\right\}$$

(note the effect of the $\gamma^0$ in the $\bar{u}$)!

$$= \frac{E+M}{2M}\begin{pmatrix}(\phi^1\phi^{1+}+\phi^2\phi^{2+}) & -(\phi^1\phi^{1+}+\phi^2\phi^{2+})\frac{\sigma.p}{E+M}\\ (\phi^1\phi^{1+}+\phi^2\phi^{2+})\frac{\sigma.p}{E+M} & -\frac{\sigma.p}{E+M}(\phi^1\phi^{1+}+\phi^2\phi^{2+})\frac{\sigma.p}{E+M}.\end{pmatrix}$$

But $(\phi^1\phi^{1+}+\phi^2\phi^{2+})_{\alpha\beta}$ is a $2\times2$ matrix

$$\equiv (\phi^1_\alpha\phi^{1+}_\beta + \phi^2_\alpha\phi^{2+}_\beta)$$

but since $\phi^1_\alpha = \begin{pmatrix}1\\0\end{pmatrix}$ $\phi^2_\alpha = \begin{pmatrix}0\\1\end{pmatrix}$

$$\phi^1_\alpha \phi^{1\dagger}_\beta + \phi^2_\alpha \phi^{2\dagger}_\beta$$

$$= \begin{pmatrix}1\\0\end{pmatrix}_\alpha (1\ 0)_\beta + \underline{\begin{pmatrix}0\\1\end{pmatrix}_\alpha (0\ 1)_\beta}$$

$$= \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} + \underline{\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix} = \mathbb{1}$$

Thus the matrix becomes.

$$\frac{E+m}{2m}\begin{pmatrix}\mathbb{1} & -\dfrac{\sigma.p}{E+m}\\ \dfrac{\sigma.p}{E+m} & -\dfrac{(\sigma.p)^2}{(E+m)^2}\end{pmatrix}$$

But the bottom right hand term

$$-\frac{(\sigma.p)^2}{(E+m)^2} = -\frac{p^2}{(E+m)^2}\mathbb{1} = -\frac{E^2-m^2}{(E+m)^2}\mathbb{1}$$

$$= -\frac{(E+m)(E-m)}{(E+m)(E+m)}\mathbb{1} = -\frac{E-m}{E+m}\mathbb{1}.$$

$\therefore$ The matrix is $\dfrac{1}{2m}\begin{pmatrix}(E+m)\mathbb{1} & -\sigma.p\\ \sigma.p & -(E-m)\mathbb{1}\end{pmatrix}$

since $\gamma^{\mu\neq 0} = \begin{pmatrix} 0 & -\sigma^{\mu} \\ \sigma^{\mu} & 0 \end{pmatrix}$ $\quad \gamma^{0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

That the matrix can be written as.

$$\frac{1}{2m}\left(\gamma^{\mu} p_{\mu} + m\,\mathbb{1}\right)_{\beta\gamma}$$

$$= \frac{1}{2m}\left(\not{p} + m\,\mathbb{1}\right)_{\beta\gamma}$$

$$\equiv \boxed{\Lambda^{+}_{\beta\gamma}(p_i) \quad \text{The positive energy projection operator at } p_i .}$$

So substitution of this result into S gives.

$$S = \frac{1}{2} \sum \bar{u}^{f}_{\alpha}(p_f, \lambda_f)\, \gamma^{0}_{\alpha\beta}\, \Lambda_{\beta\gamma}(p_i)\, \gamma^{0}_{\gamma\delta}\, u_{\delta}(p_f, \lambda_f)$$

but remember $u_{\delta}(p_f, \lambda_f)$ is just a number which can be moved within the expression.

$$S = \frac{1}{2} \underbrace{\sum_{f} u^{f}_{\delta}(p_f, \lambda_f)\, \bar{u}^{f}_{\alpha}(p_f, \lambda_f)}_{\Lambda^{+}_{\delta\alpha}(p_f)}\, \gamma^{0}_{\alpha\beta}\, \Lambda_{\beta\gamma}\, \gamma^{0}_{\gamma\delta} .$$

So

$$S = \frac{1}{2} \Lambda^{+}_{\delta\alpha}(p_f) \gamma^{0}_{\alpha\beta} \Lambda^{+}_{\beta\gamma}(p_i) \gamma^{0}_{\gamma\delta}$$

but if we consider all the contractions over $\alpha, \beta$ and $\gamma$ first we are left with a term in a matrix $M_{\delta\delta} \equiv$ The trace of the resultant matrix.

$$\therefore \quad S = \frac{1}{2} T_r \left\{ \Lambda^{+}(p_f) \gamma^{0} \Lambda(p_i) \gamma^{0} \right\}.$$

Thus all such terms in S can be immediately written down as the trace of a matrix composed from products of $\gamma$ matrices:

eg.

$$S = \frac{1}{2} T_r \left\{ (\gamma^{\mu} p_{\mu} + m \mathbb{1}) \gamma^{0} (\gamma^{\nu} p_{\nu} + m \mathbb{1}) \gamma^{0} \right\}.$$

We will now consider theorem that will help us quickly evaluate such traces.

# $\gamma$ Matrix trace theorems

Before we derive these theorems we will introduce a new matrix $\gamma^5$.

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$$

this matrix anticommutes with the other $\gamma$ matrices.

$$\gamma^5\gamma^\mu + \gamma^\mu\gamma^5 = 0 \qquad (\gamma^5)^2 = \mathbb{1}.$$

## Trace theorems:

1) $\mathrm{Tr}(\not{a}\not{b}\not{c}\ldots\ldots) = \mathrm{Tr}(\not{a}\not{b}\not{c}\,\gamma^5\gamma^5)$

taking the transpose

$$= \mathrm{Tr}(\gamma^5\not{a}\not{b}\not{c}\ldots\gamma^5)$$

we can now flip the right hand $\gamma^5$ back through the terms but it will generate a new factor of $(-1)$ each time.

$$= (-1)^n\, \mathrm{Tr}(\gamma^5\gamma^5\not{a}\not{b}\not{c}\ldots).$$

if there are an odd number of terms.

$$= (-1)\,\mathrm{Tr}(\not{a}\not{b}\not{c})$$

Thus the trace of the product of an odd number of gamma matrices is 0.

$$= 0.$$

2) $Tr(\mathbb{1}) = 4$ since $\mathbb{1}$ in Dirac's equation is $4\times4$.

3) $Tr(\not{a}\not{b}) = Tr(a_\mu \gamma^\mu b_\nu \gamma^\nu)$

$$= Tr(a_\mu b_\nu \gamma^\mu \gamma^\nu)$$

but since $Tr(A) = Tr(\tilde{A})$

and $\widetilde{\gamma^\mu \gamma^\nu} = \tilde{\gamma}^\nu \tilde{\gamma}^\mu = (-\gamma^\nu)(-\gamma^\mu) = \gamma^\nu \gamma^\mu$

we can add a $\gamma^\nu \gamma^\mu$ term to the trace.

$$= \frac{1}{2} Tr(a_\mu b_\nu (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu))$$

$$= \frac{1}{2} Tr(a_\mu b_\nu 2 g^{\mu\nu} \mathbb{1})$$

$$= Tr(a_\mu b_\nu g^{\mu\nu} \mathbb{1})$$

$$= a_\mu b_\nu g^{\mu\nu} Tr(\mathbb{1})$$

$$= \underline{4 a_\mu b^\mu}$$

Another way of expressing the same information

$$\boxed{Tr(\gamma^\mu \gamma^\nu) = g^{\mu\nu} Tr(\mathbb{1}) = 4 g^{\mu\nu}}$$

4) $$\mathrm{Tr}(\gamma^5) = \mathrm{Tr}(\underbrace{\gamma^0\gamma^0}_{\mathbb{1}}\gamma^5)$$

$$= -\mathrm{Tr}(\gamma^0\gamma^5\gamma^0)$$ $\gamma^5\gamma^0$ anticommute

---

But since

$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$$

Proof $\mathrm{Tr}(AB) = A_{\alpha\beta}B_{\beta\alpha} = B_{\beta\alpha}A_{\alpha\beta}$ $\left\{\begin{array}{l}\text{Since } A_{\alpha\beta} \\ B_{\alpha\beta} \text{ are} \\ \text{just} \\ \text{numbers}\end{array}\right.$

$$= \mathrm{Tr}(BA)$$

This also implies

$$\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB)$$ – treating $(AB)$ as a single matrix

$$= \mathrm{Tr}(BCA)$$

Matrix cyclic condition.

---

Thus $$-\mathrm{Tr}(\gamma^0\gamma^5\gamma^0)$$

$$= -\mathrm{Tr}(\gamma^0\gamma^0\gamma^5)$$

$$= -\mathrm{Tr}(\gamma^5)$$

But $\gamma^5 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ – true.

Thus $\mathrm{Tr}(\gamma^5) = 0$.

N.B. This proof could be repeated for all $\gamma$ matrices – but of course they are all traceless.

5) $Tr(\gamma^5 \not{a} \not{b}) = Tr(a_\mu b_\nu \gamma^5 \gamma^\mu \gamma^\nu)$

$= a_\mu b_\nu Tr(\gamma^5 \gamma^\mu \gamma^\nu)$

But let us consider each of the traces in the sum separately.

$$Tr(\gamma^5\gamma^\mu\gamma^\nu) = Tr(\gamma^\alpha\gamma^\alpha\gamma^5\gamma^\mu\gamma^\nu) \quad \alpha = 0,\ \alpha \neq \mu, \nu.$$

$$= -Tr(\gamma^\alpha\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha)$$

$$= -Tr(\gamma^\alpha\gamma^\alpha\gamma^5\gamma^\mu\gamma^\nu)$$

$$= -Tr(\gamma^5\gamma^\mu\gamma^\nu)$$

$$= 0. \qquad \boxed{\text{as } \gamma^{i2} = -\mathbb{1} \quad i \neq 0.}$$

$$Tr(\gamma^5\gamma^\mu\gamma^\nu) = -Tr(\gamma^\alpha\gamma^\alpha\gamma^5\gamma^\mu\gamma^\nu) \quad \alpha \neq 0,\ \alpha \neq \mu,\nu.$$

$$= Tr(\gamma^\alpha\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha)$$

$$= Tr(\gamma^\alpha\gamma^\alpha\gamma^5\gamma^\mu\gamma^\nu)$$

$$= -Tr(\gamma^5\gamma^\mu\gamma^\nu).$$

$$= 0.$$

Thus wether $\mu$ or $\nu = 0$ the

$$Tr(\gamma^5\gamma^\mu\gamma^\nu) = 0.$$

Thus all terms in the sum $a_\mu b_\nu Tr(\gamma^5 \gamma^\mu \gamma^\nu)$ are $\phi$ $\therefore$ $\boxed{Tr(\gamma^5 \not{a} \not{b}) = 0}$

6) $Tr(\gamma^5 \not{a} \not{b} \not{c} \not{d}) = a_\nu b_\mu c_\rho d_\sigma \, Tr(\gamma^5 \gamma_\nu \gamma_\mu \gamma_\rho \gamma_\sigma)$

Now considering the individual terms again.

$$Tr(\gamma^5 \gamma_\nu \gamma_\mu \gamma_\rho \gamma_\sigma) = Tr(\gamma_\alpha \gamma_\alpha \gamma^5 \gamma_\nu \gamma_\mu \gamma_\rho \gamma_\sigma)$$

so long as $\nu \mu \rho \sigma$ do not contain all of $0\,1\,2\,3$.

$\therefore$ as before

$$= 0.$$

Now let us suppose $\nu \mu \rho \sigma = 0\,1\,2\,3$.

$$Tr(\gamma^5 \gamma_\nu \gamma_\mu \gamma_\rho \gamma_\sigma) = Tr(\gamma^5 \gamma^5)$$

$$= Tr(\mathbb{1}) = -i4.$$

note too that if we have or alter the permutation by one interchange we change the sign. The sign is given by $\varepsilon_{\nu\mu\rho\sigma}$

$$= i4 \, \varepsilon_{\nu\mu\rho\sigma}$$

where $\varepsilon_{\nu\mu\rho\sigma} = \varepsilon_{0123}$

Thus

$$\boxed{Tr(\gamma^5 \not{a} \not{b} \not{c} \not{d}) = i4 \, a_\nu b_\mu c_\rho d_\sigma \, \varepsilon_{\nu\mu\rho\sigma}.}$$

7) $$Tr(\not{a}_1 \dots \not{a}_n) = 2a_1 \cdot a_2 \, Tr(\not{a}_3 \dots \not{a}_n) - Tr(\not{a}_2 \not{a}_1 \dots \not{a}_n)$$

Since $(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = 2g^{\mu\nu} \mathbb{1}$.

$$\therefore \gamma^\mu \gamma^\nu = 2g^{\mu\nu} \mathbb{1} - \gamma^\nu \gamma^\mu.$$

and since $Tr(a^1_\mu a^2_\nu g^{\mu\nu} \not{a}^3 \dots \not{a}^n) = a^{1\mu} a^2_\mu Tr(\not{a}^3 \dots \not{a}_n)$

But taking the right hand term

$$Tr(\not{a}_2 \not{a}_1 \not{a}_3 \dots \not{a}_n) = -2(a_1 \cdot a_3) \, Tr(\not{a}_2 \not{a}_4 \dots \not{a}_n) - Tr(\not{a}_2 \not{a}_3 \not{a}_1 \dots \not{a}_n)$$

and so on until until

$$Tr(\not{a}_2 \not{a}_3 \dots \not{a}_1 \not{a}_n) = 2(a_1 \cdot a_n) Tr(\not{a}_2 \not{a}_3 \dots \not{a}_n) \underbrace{- Tr(\not{a}_2 \dots \not{a}_n \not{a}_1)}_{Tr(\not{a}_1 \not{a}_2 \dots \not{a}_n)}$$

which can be taken over the L.h.s.

$$\therefore 2Tr(\not{a}_1 \not{a}_2 \dots \not{a}_n) = 2(a_1 \cdot a_2) \, Tr(\not{a}_3 \dots \not{a}_n) - 2(a_1 \cdot a_3) \, Tr(\not{a}_2 \not{a}_4 \dots \not{a}_n).$$

$\dots\dots$

$$\therefore Tr(\not{a}_1 \not{a}_2 \dots \not{a}_n) = \cancel{2}(a_1 \cdot a_2) \, Tr(\not{a}_3 \not{a}_4 \dots \not{a}_n) - (a_1 \cdot a_3) \dots\dots$$

if $n = 4$.

$$
\begin{aligned}
Tr(\not a_1 \not a_2 \not a_3 \not a_4) &= (a_1 . a_2) . Tr(\not a_3 \not a_4) \\
&- (a_1 . a_3)\, Tr(\not a_2 \not a_4) \\
&+ (a_1 . a_4)\, Tr(\not a_2 \not a_3) .
\end{aligned}
$$

$$
\begin{aligned}
= 4\big[ &(a_1 . a_2)(a_3 . a_4) \\
&- (a_1 . a_3)(a_2 . a_4) \\
&+ (a_1 . a_4)(a_2 . a_3) \big]
\end{aligned}
$$

so higher even values of $n$ can be solved by substituting $Tr(\not a_1 \not a_2 \not a_3 \not a_4)$ etc.

Another way of expressing this is

$$
\boxed{
\begin{aligned}
Tr(\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma) = 4(&g^{\mu\nu} . g^{\rho\sigma} \\
&- g^{\mu\rho} . g^{\nu\sigma} \\
&+ g^{\mu\sigma} . g^{\nu\rho}) .
\end{aligned}
}
$$