$p=0$. Spin

We saw that the hamiltonian

$$\hat{H} = \begin{pmatrix} m\mathbb{1} & 0 \\ 0 & -m\mathbb{1} \end{pmatrix}$$

and the operator

$$\Sigma_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

commutes with this so can be used to label the degeneracy. Note however that the positive energy states have just two different eigenvalues

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-iEt} \qquad \psi_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-iEt}$$

If this is to be interpreted as the z-component of a spin the only spin state that would have only two values is $1/2$.

Thus $S_3 = \frac{1}{2}\Sigma_3 = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$ ∴ eigenvalues $S_3 = \pm\frac{1}{2}$

can the angular momentum 3 component be extrapolated to the other two components

ie $\hat{\underline{S}} = \frac{1}{2}\underline{\Sigma} = \frac{1}{2}\begin{pmatrix} \underline{\sigma} & 0 \\ 0 & \underline{\sigma} \end{pmatrix}$

$$\therefore \hat{\underline{S}} = \frac{1}{2}\underline{\Sigma} = \frac{1}{2}\left[\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}\hat{\underline{i}} + \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix}\hat{\underline{j}} + \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}\hat{\underline{k}}\right]$$

$$\therefore \quad \hat{S}^2 = \frac{1}{4} \cdot 3 \; \mathbb{1}$$

which can be written as

$$\hat{S}^2 = \frac{1}{2}\left(\frac{1}{2} + 1\right) \mathbb{1}$$

note also $[\hat{S}^2, S_3] = 0$.
[so $\hat{S}^2$ can also be used to label the state]

Thus the states at rest can be interpreted as states of $\frac{1}{2}$-spin. whereas orbital angular momentum comes in packets of $\hbar$ spin states are only $\pm 1/2$.

$p \neq 0$ helicity

The hamiltonian is now more complex

$$\hat{H} = \alpha . p + \beta m = \begin{pmatrix} 0 & \sigma . p \\ \sigma . p & 0 \end{pmatrix} + m \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

$$= \begin{pmatrix} m\mathbb{1} & \sigma . p \\ \sigma . p & -m\mathbb{1} \end{pmatrix}$$

$\underline{\Sigma}$ no longer commutes with $\hat{H}$. However

lets consider $$h(\underline{p}) = \begin{pmatrix} \frac{\underline{\sigma} . \underline{p}}{|p|} & 0 \\ 0 & \frac{\underline{\sigma} . \underline{p}}{|p|} \end{pmatrix}$$

which we can see reflects some of the structure of $\hat{H}$

$$\begin{pmatrix} \dfrac{\sigma\cdot p}{|p|} & 0 \\ 0 & \dfrac{\sigma\cdot p}{|p|} \end{pmatrix} \begin{pmatrix} M\,\mathbb{1} & \sigma\cdot p \\ \sigma\cdot p & -M\,\mathbb{1} \end{pmatrix} = \begin{pmatrix} \dfrac{M\,\sigma\cdot p}{|p|} & \dfrac{(\sigma\cdot p)(\sigma\cdot p)}{|p|} \\ \dfrac{(\sigma\cdot p)(\sigma\cdot p)}{|p|} & -\dfrac{M\,\sigma\cdot p}{|p|} \end{pmatrix}$$

$$\begin{pmatrix} M\,\mathbb{1} & \sigma\cdot p \\ \sigma\cdot p & -M\,\mathbb{1} \end{pmatrix} \begin{pmatrix} \dfrac{\sigma\cdot p}{|p|} & 0 \\ 0 & \dfrac{\sigma\cdot p}{|p|} \end{pmatrix} = \begin{pmatrix} \dfrac{M\,\sigma\cdot p}{|p|} & \dfrac{(\sigma\cdot p)(\sigma\cdot p)}{|p|} \\ \dfrac{(\sigma\cdot p)(\sigma\cdot p)}{|p|} & \dfrac{M\,\sigma\cdot p}{|p|} \end{pmatrix}$$

Thus

$$\hat{h}(p) = \begin{pmatrix} \dfrac{\sigma\cdot p}{|p|} & 0 \\ 0 & \dfrac{\sigma\cdot p}{|p|} \end{pmatrix}$$

can be used to label the moving particle $(p \neq 0)$ states.

Let us write the Dirac in the 2 component block form.

$$p^0 \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} m\,\mathbb{1} & (\sigma.p) \\ (\sigma.p) & -m\,\mathbb{1} \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix}$$

Thus

$$p^0 \phi = m \phi \mathbb{1} + (\sigma.p)\chi \qquad -(1)$$

$$p^0 \chi = (\sigma.p)\phi - m\,\mathbb{1}\,\chi \qquad -(2)$$

Using (2)

$$(p^0 + m)\chi = (\sigma.p)\phi$$

$$\Rightarrow \quad \chi = \left(\frac{\sigma.p}{p^0+m}\right)\phi$$

Substituting this into (1)

$$p^0 \phi = \left[m\,\mathbb{1} + \left(\frac{(\sigma.p)(\sigma.p)}{p^0+m}\right)\right]\phi$$

But $\quad \sigma_i \sigma_j = \varepsilon_{ijk} \sigma_k + \mathbb{1} \delta_{ij}$

Thus $\quad a_i \sigma_i b_j \sigma_j = a_i b_j \sigma_i \sigma_j$

$$= a_i b_j \{ \varepsilon_{ijk} \sigma_k + \mathbb{1} \delta_{ij} \}$$

$$= a_i b_j \delta_{ij} \mathbb{1} + a_i b_j \varepsilon_{ijk} \sigma_k$$

$$= \underline{a} \cdot \underline{b} \, \mathbb{1} + (\underline{a} \times \underline{b}) \cdot \underline{\sigma}$$

Thus $\quad (\sigma \cdot p)(\sigma \cdot p) = p^2 \mathbb{1}.$

So $\quad p^0 \phi = m \mathbb{1} \phi + \dfrac{p^2 \mathbb{1} \phi}{p^0 + m}$

$$\Rightarrow \quad (p^0 - m) \phi = \frac{p^2}{p^0 + m} \mathbb{1} \phi$$

$$p^{02} - m^2 \phi = p^2 \phi$$

$$\therefore \quad \boxed{p^{02} = p^2 + m^2}$$

So now we know too that the spinor takes a block form and the block are related

the positive energy solutions can be written as

$$\omega^{1,2} \quad \exp -i(p^\mu x_\mu)$$

where $\omega^{1,2} = \begin{pmatrix} \phi^{1,2} \\ \chi^{1,2} \end{pmatrix}$

and $\chi^{1,2} = \dfrac{\sigma . p}{E+M} \phi^{1,2}$

where $\phi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\qquad \phi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} \phi^{1,2} \\ \dfrac{\sigma . p}{E+m} \phi^{1,2} \end{pmatrix} e^{-ip^\mu x_\mu}$$

N.B.

for $p = 0$

$\omega^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad \omega^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$

$\phi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \phi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

the negative energy solution can be written as

$$\omega^{3,4} \exp i(p^\mu x_\mu)$$

N.B for $p=0$

$$\omega^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \omega^4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\chi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\omega^{3,4} = \begin{pmatrix} \phi^{1,2} \\ \chi^{1,2} \end{pmatrix}$$

and $$\phi^{1,2} = \frac{\sigma.p}{E+m} \chi^{1,2}$$

where $\chi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\chi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$= \begin{pmatrix} \left(\frac{\sigma.p}{E+m}\right) \chi^{1,2} \\ \chi^{1,2} \end{pmatrix} e^{ip^\mu x_\mu}$$

So what are the eigenvalues of the helicity operator.

$$\begin{pmatrix} \frac{\sigma.p}{|p|} & 0 \\ 0 & \frac{\sigma.p}{|p|} \end{pmatrix} \begin{pmatrix} \phi^{\pm} \\ \frac{\sigma.p}{E+m}\phi^{\pm} \end{pmatrix} = \lambda^{\pm} \begin{pmatrix} \phi^{\pm} \\ \frac{\sigma.p}{E+m}\phi^{\pm} \end{pmatrix}$$

The first term gives $\frac{\sigma.p}{E+m} \phi^{\pm} = \lambda^{\pm} \phi^{\pm}$

which can be written as

$$\frac{1}{|p|} \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

which can be solved using the characteristic equation

$$\frac{1}{|p|} \begin{vmatrix} p_z - \lambda|p| & p_x - ip_y \\ p_x + ip_y & -p_z - \lambda|p| \end{vmatrix} = 0.$$

$$\therefore (p_z - \lambda|p|)(-p_z - \lambda|p|) - (p_x - ip_y)(p_x + ip_y) = 0.$$

$$\therefore -p_z^2 + \lambda^2|p| - p_x^2 - p_y^2 = 0.$$

$$\therefore \lambda^2 = \frac{p_x^2 + p_y^2 + p_z^2}{|p|^2}$$

$$\therefore \boxed{\lambda = \pm 1}$$

To find the eigen vectors for $\lambda_+ = +1$.

$$\begin{pmatrix} \frac{P_z}{|P|} - 1 & \frac{P_x - iP_y}{|P|} \\ \frac{P_x + iP_y}{|P|} & \frac{-P_z}{|P|} - 1 \end{pmatrix} \begin{pmatrix} x_1^+ \\ x_2^+ \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

1) $$\left(\frac{P_z}{|P|} - 1\right) x_1^+ + \left(\frac{P_x - iP_y}{|P|}\right) x_2^+ = 0.$$

2) $$\left(\frac{P_x + iP_y}{|P|}\right) x_1^+ + \left(\frac{-P_z}{|P|} - 1\right) x_2^+ = 0.$$

from 1) $$x_1^+ = \frac{\left(\frac{P_x - iP_y}{|P|}\right)}{\left(1 - \frac{P_z}{|P|}\right)} x_2^+$$

$$x_1^+ = \frac{P_x - iP_y}{|P| - P_z} x_2^+$$

2) $$x_2^+ = \frac{P_x + iP_y}{|P| + P_z} x_1^+$$

We then chose a value of x_1^+ or x_2^+ that Normalizes the eigen vects.

Thus finally.

$$\omega(p, \lambda \pm 1) = \begin{pmatrix} \phi^{\pm} \\ \dfrac{\sigma.p}{E+m}\,\phi^{\pm} \end{pmatrix}.$$

The positive and negative helicity states.

The DIRAC particle current:

$$(1) \quad -i\frac{\partial \psi}{\partial t} = \left(+i\,\alpha.\nabla + \beta m\right)\psi$$

An taking the complex conjugate and transpose

$$(2) \quad i\frac{\partial \tilde{\psi}^*}{\partial t} = -i\,\widetilde{\alpha.\nabla\psi}^* + \widetilde{\beta m\psi}^*$$

but since $\widetilde{(AB)}^* = \tilde{B}^*\tilde{A}^*$

$$i\frac{\partial \tilde{\psi}^*}{\partial t} = -i\,\nabla\tilde{\psi}^*.\tilde{\alpha}^* + \tilde{\psi}^*\tilde{\beta}^*m$$

but α and β are hermitian $\therefore$ $\tilde{\alpha}^* = \alpha^+ = \alpha$

$\tilde{\beta}^* = \beta^+ = \beta$.

$$\Rightarrow \quad i\frac{\partial \tilde{\psi}^*}{\partial t} = -i\,\nabla\tilde{\psi}^*\alpha + \tilde{\psi}^*\beta m.$$

multiplying (1) by $\psi^+ = \tilde{\psi}^*$ and pre multiplying (2) by ψ and subtracting we

$$-i\left(\tilde{\psi}^*\frac{\partial \psi}{\partial t} + \psi\frac{\partial \tilde{\psi}^*}{\partial t}\right) = i\left(\tilde{\psi}^*\alpha.\nabla\psi + \psi\,\alpha.\nabla\tilde{\psi}^*\right)$$

$$\Rightarrow \quad \boxed{-i\frac{\partial \psi^+\psi}{\partial t} = i\nabla.\left(\psi^+\alpha\psi\right)}.$$

Thus the Dirac current is

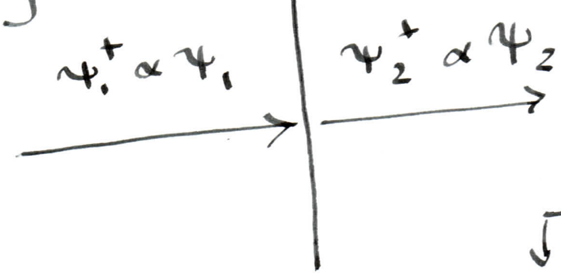
$$j_\mu = (\psi^\dagger \psi , \psi^\dagger \underline{\alpha} \psi)$$

which satisfies the usual equation (flux equation).

$$\partial^\mu j_\mu = 0 .$$

Thus $\psi^\dagger\psi$ is associated with the probability and $\psi^\dagger \alpha \psi$ the flux.

So N.B. There is <u>no</u> gradient in the flux so for spin 1/2 particles only the wave function needs be continuous across a boundary



$\therefore \psi_1 = \psi_2$.

space part

Also rewriting the ψ as $\phi \underline{u}$

spinor

$$j^\mu = (\phi^* \phi \, \underline{u}^\dagger \underline{u} , \phi^* \phi \, \underline{u}^\dagger \underline{\alpha} \, \underline{u})$$

Normalization of the Dirac Wave function

Since $\psi^\dagger\psi$ is the zeroth component of the Dirac particle 4-vector or flux we associate this quantity with the probability

$$\psi^\dagger\psi = \underbrace{\alpha^*\alpha}_{\substack{\text{spacial}\\ \text{wave function.}}} u^\dagger u$$

$$\alpha^*\alpha \sim L^3$$

$$u^\dagger u = \begin{pmatrix} \phi' \\ \frac{\sigma.p}{E+M}\phi' \end{pmatrix}^\dagger \begin{pmatrix} \phi' \\ \frac{\sigma.p}{E+M}\phi' \end{pmatrix}$$

in terms of components

$$\phi_i^{\dagger} \mathbb{1}_{ij} \phi_j' + \phi_i^{\dagger\prime} \left(\frac{\sigma.p}{E+M}\right)^2_{ij} \phi_j'$$

$$= \phi_i^{\dagger\prime} \mathbb{1}_{ij} \phi_j + \phi_i^{\dagger\prime} \frac{p^2}{E+M} \mathbb{1}_{ij} \phi_j'$$

$$= \left(1 + \frac{p^2}{(E+M)^2}\right)\left(\phi_i^{\dagger\prime} \mathbb{1}_{ij} \phi_j\right).$$

$$= \frac{E^2+M^2+2EM+p^2}{(E+M)^2} \phi_i^{\dagger\prime} \mathbb{1}_{ij} \phi_j'$$

$$= \frac{2E(E+M)}{(E+M)^2} \underbrace{\phi_i^{\dagger\prime} \mathbb{1}_{ij} \phi_j'}_{1} \quad - \text{ if } \phi' = \begin{pmatrix}1\\0\end{pmatrix} \text{ or } \begin{pmatrix}0\\1\end{pmatrix}$$

$$= \frac{2E}{E+M}$$

$$\Sigma_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\psi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-iEt} \qquad \psi_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-iEt}$$

$$\Sigma_3 |\psi_1\rangle = + |\psi_1\rangle \qquad \Sigma_3 |\psi_2\rangle = - |\psi_2\rangle$$

<u>spin:</u>

Let us define

$$S_z = \tfrac{1}{2}\hbar \Sigma_3 = \tfrac{1}{2}\hbar \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

$$S_x = \tfrac{1}{2}\hbar \Sigma_x = \tfrac{1}{2}\hbar (\Sigma_1 + i\Sigma_2)$$

$$= \tfrac{1}{2}\hbar \begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix}$$

NB.

$\sigma_z = \sigma_3$

$\sigma_x = \sigma_1 + i\sigma_2$

$\sigma_y = \sigma_1 + i\sigma_2$.

$$S_y = \tfrac{1}{2}\hbar \Sigma_y = \tfrac{1}{2}\hbar (\Sigma_1 - i\Sigma_2)$$

$$= \tfrac{1}{2}\hbar \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix}$$

$$\underline{S} = \tfrac{1}{2}\hbar \underline{\Sigma} = \tfrac{1}{2}\hbar (\Sigma_x \hat{\underline{i}} + \Sigma_y \hat{\underline{j}} + \Sigma_k \hat{\underline{k}})$$

$$= \tfrac{1}{2}\hbar \left[\begin{pmatrix} \sigma_x & 0 \\ 0 & \sigma_x \end{pmatrix} \hat{\underline{i}} + \begin{pmatrix} \sigma_y & 0 \\ 0 & \sigma_y \end{pmatrix} \hat{\underline{j}} + \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} \hat{\underline{k}} \right]^2$$

$$= \tfrac{1}{4}\hbar^2 \left[\begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \hat{\underline{i}} + \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \hat{\underline{j}} + \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} \hat{\underline{k}} \right]$$

$$= \tfrac{3}{4}\hbar \, \mathbb{1}.$$

$$= \tfrac{1}{2}(\tfrac{1}{2}+1)\hbar \, \mathbb{1}.$$

$$\boxed{\therefore \hat{S}^2 \psi = \tfrac{1}{2}(\tfrac{1}{2}+1)\psi}$$

This means ψ can be thought of as particle spin $1/2$.

So how do we want to normalize the $u^\dagger u$.

1) Covariant choice

$$u_i^\dagger u_j = \delta_{ij} \frac{E}{M}$$

Why do this, because this transforms like E and is therefore manifestly like a 0th component of a four vector. Thus if we normalize the spinor in this way we get

$$u_i = \left(\frac{E+M}{2M}\right)^{1/2} \begin{pmatrix} \phi^i \\ \frac{\sigma . p}{E+M} \phi^i \end{pmatrix}$$

The convention is that u automatically implies the covariant normalization is included $(E+M/2M)^{1/2}$.

ii) Non-covariant choice.

If we want the probabilistic interpretation we need $u_i^\dagger u_j = \frac{1}{L^3}\delta_{ij}$

This clearly mean the 4 current is no longer manifestly covariant but mean we can interpret in terms of particles.

$$\boxed{\psi^i = \frac{1}{L^{3/2}}\left(\frac{M}{E}\right)^{1/2} u(p,i)\, e^{-ip^\mu x_\mu}.}$$

N.B. u has covariant normalization built in.

If we use this normalization

$$\boxed{j = \psi^\dagger \alpha \psi = \frac{v}{L^3}}.$$