Dirac wondered if the problem with negative energy solutions could be avoided if the energy-momentum equation could be made linear in energy.

So he tryed to find a form:-

$$i\hbar \frac{\partial}{\partial t} \psi(x,t) = -i\hbar \left\{ \alpha^1 \frac{\partial}{\partial x^1} + \alpha^2 \frac{\partial}{\partial x^2} + \alpha^3 \frac{\partial}{\partial x^3} \right\} \psi + \beta m \psi .$$

This can be written as

$$\boxed{E = \underline{\alpha} \cdot \underline{p} + \beta m}.$$

{where $\alpha^1 \alpha^2 \alpha^3 \beta$ are quantities to be determined

however this cannot change the fact that ultimately we have to recover the relationship

$$\boxed{E^2 = p^2 + m^2}$$

So what does that imply for $\underline{\alpha}$ and β.

$$E^2 = (\underline{\alpha}.\underline{p})^2 + m(\underline{\alpha}\beta + \beta\underline{\alpha}).\underline{p} + \beta^2 m^2$$

$$\Rightarrow E^2 = (\alpha_i p_i)(\alpha_j p_j) + m(\underline{\alpha}\beta + \beta\underline{\alpha}).p + \beta^2 m^2$$

we can immediately write down some of the properties of $\underline{\alpha}\beta$.

1) $\alpha_i^2 = \mathbb{1}$

2) $\alpha_i\alpha_j + \alpha_j\alpha_i = \{a_i\alpha_j\} = 0$ anticommutates

3) $\alpha_i\beta + \beta\alpha_i = \{\underline{\alpha}\beta\} = 0$ anti commutes

4) $\beta^2 = \mathbb{1}$

so the $\underline{\alpha}$ and β anticommute and are self inverse.

Further Properties:

All quantum mechanical operator producing real observable quantites are hermitian.

$$E|\psi_i\rangle = \lambda_i|\psi_i\rangle \quad ; \quad E|\psi_j\rangle = \lambda_j|\psi_j\rangle$$

$$\Rightarrow \langle\psi_j|E|\psi_i\rangle = \lambda_i\langle\psi_j|\psi_i\rangle \quad ; \quad \langle\psi_i|E|\psi_j\rangle = \lambda_j\langle\psi_i|\psi_j\rangle$$

lets take the transpose of the complex conjugate of the L.h.s.

$$\langle\psi_j|E|\psi_i\rangle^* = \lambda_i^*\langle\psi_i|\psi_j\rangle = \langle\psi_i|\tilde{E}^*|\psi_j\rangle$$

L.h.s - r.h.s.

$$\therefore \langle\psi_i|\tilde{E}^*|\psi_j\rangle - \langle\psi_i|E|\psi_j\rangle = \lambda_i^* - \lambda_j\langle\psi_i|\psi_j\rangle$$

$$\langle \psi_i | E^{+} | \psi_j \rangle - \langle \psi_i | E | \psi_j \rangle = (\lambda_i^{*} - \lambda_j) \langle \psi_i | \psi_j \rangle$$

since $E^{+} = \tilde{E}^{*}$

<u>if $i \neq j$ $\langle \psi_i | \psi_j \rangle = 0$</u>

$$\therefore \langle \psi_i | E^{+} | \psi_j \rangle - \langle \psi_i | E | \psi_j \rangle = 0 .$$

$\therefore$ $E^{+} = E$ thus the operator E is hermitian.

<u>if $i = j$ $\langle \psi_i | \psi_i \rangle = 1$</u>

$$\therefore (\lambda_i^{*} - \lambda_i) = 0$$

Thus the eigenvalue is real.

Thus we have proven the original equation statement.

Thus since E is a q.m. operator for a measurable quantity and

$$E \psi = (\underline{\alpha} . \underline{p} + \beta m) \psi$$

then

i) The $\underline{\alpha}$'s and β are hermitian $\alpha_i = \alpha_i^{+} = \tilde{\alpha}_i^{*}$

$\beta = \beta^{+} = \tilde{\beta}^{*}$

ii) $Tr(\alpha_i) = Tr(\beta) = 0$

where $Tr(\beta)$
$\equiv$ the sum of the leading diagonals.
$\equiv \sum \beta_{ii}$

Proof

$$\alpha^i \beta + \beta \alpha^i = 0$$

$$\alpha^i \beta . \beta + \beta \alpha^i \beta = 0$$

$$\therefore \quad \alpha^i \beta . \beta = - \beta \alpha^i \beta$$

$$\Rightarrow \quad \alpha^i = - \beta \alpha^i \beta \qquad \text{since } \beta^2 = \mathbb{1}.$$

$$Tr(\alpha_i) = - Tr(\beta \alpha_i \beta)$$

$$= - Tr(\alpha_i \beta . \beta) \qquad \text{since } Tr(AB) = Tr(BA)^{**}$$

$$\therefore \quad Tr(\alpha_i) = - Tr(\alpha_i)$$

$$\therefore \quad Tr(\alpha_i) = 0.$$

you can do the same for β

$$\Rightarrow Tr(\beta) = 0.$$

** $AB = A_{ij} B_{jk}$

$Tr(AB) = A_{ij} B_{ji} = B_{ji} A_{ij} = Tr(BA)$

iii) The eigenvalues of $\alpha_i\ \beta$ are ± 1

<u>Proof:</u> We can use the special properties of the eigenvalue equations and vectors to form a diagonal matrix.

Let A be a 2×2 matrix

$$A\,|x_i\rangle = \lambda_i\,|x_i\rangle$$

now lets consider what would happen if we constructed a matrix of the eigenvectors and sandwiched the A matrix as follows.

$$(x_1, x_2)^T A\ (x_1, x_2) = (x_1, x_2)^T (\lambda_1 x_1, \lambda_2 x_2)$$

$$= \begin{pmatrix} \lambda_1\ x_1^T x_1 & \lambda_2\ x_1^T x_2 \\ \lambda_1\ x_2^T x_1 & \lambda_2\ x_2^T x_2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

for an $(n\times n)$ matrix we would have

$$\begin{pmatrix} \lambda_i\ x_1^T x & & & \\ & \lambda_2\ x_2^T x_2 & & \\ & & \ddots & \\ & & & \lambda_n x_n^T x_n \end{pmatrix}$$

All the diagonal terms would be zero as $x^T_i\, x_i = 0$.

iii) The eigenvalues of α_i β are ± 1

Proof: We can use the special properties of the eigenvalue equations and vectors to form a diagonal matrix.

Let A be a 2×2 matrix

$$A|x_i\rangle = \lambda_i |x_i\rangle$$

now lets consider what would happen if we constructed a matrix of the eigenvectors and sandwiched the A matrix as follows.

$$(x_1, x_2)^T A (x_1, x_2) = (x_1, x_2)^T (\lambda_1 x_1, \lambda_2 x_2)$$

$$= \begin{pmatrix} \lambda_1 \, x_1^T x_1 & \lambda_2 \, x_1^T x_2 \\ \lambda_1 \, x_2^T x_1 & \lambda_2 \, x_2^T x_2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

for an $(n\times n)$ matrix we would have

$$\begin{pmatrix} \lambda_1 x_1^T x & & & \\ & \lambda_2 x_2^T x_2 & & \\ & & \ddots & \\ & & & \lambda_n x_n^T x_n \end{pmatrix}$$

All the diagonal terms would be zero as $x^T_i x_j = 0$.

but since $\alpha_i^+ \alpha_i = \alpha_i^2 = \mathbb{1}$.

$$= U^+ \alpha_i^+ \alpha_i U = U^T \alpha_i^2 U = U^T \mathbb{1} U = \mathbb{1}$$

$$= U^+ \alpha_i^+ \underbrace{U U^+}_{\mathbb{1}} \alpha_i U = \mathbb{1}$$

Thus since

$$U^+ \alpha_i U = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_4 \end{pmatrix}$$ "diagonalized matrix"

and $\overbrace{U^+ \alpha_i^+ U}^{*} = U^+ \widetilde{\left(U^+ \alpha_i^+\right)}^{*}$

$$= U^+ (\alpha_i) U = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_4 \end{pmatrix}$$

Thus $U^+ \alpha_i^+ U = \overbrace{\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}}^{*} = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$

Thus $\begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_4 \end{pmatrix} \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_4 \end{pmatrix}$

$$= \begin{pmatrix} \lambda_1^2 & & & 0 \\ & \lambda_2^2 & & \\ & & \ddots & \\ 0 & & & \lambda_n^2 \end{pmatrix}$$

Thus since $\alpha_i^2 = \mathbb{1}$. and $\beta^2 = \mathbb{1}$.

$$\lambda_i^2 = 1 \quad \therefore \quad \boxed{\lambda_i = \pm 1. \quad \text{for } \alpha_i \text{ and } \beta}$$

iv) α^i and β have even dimensionality. (8)

Proof:

We have shown

$$Tr\,(U^{+}\alpha_i U) = Tr\,(a_i\, U U^{+}) = Tr\,(a_i)$$

that the sum of the Trace of a square sym. matrix is $(\sum_i \lambda_i)$ the sum of the eigen values.

But from a separate argument

$$Tr(\alpha_i) = Tr(\beta) = 0.$$

so the only way $\sum_i \lambda_i = 0$ for $\lambda_i = \pm 1$ is for there to be an <u>even</u> number of them.

So what can the α_i, β be?

The most general 2×2 hermitian matrix $H = \tilde{H}^* = H^\dagger$

is $\begin{pmatrix} a & b \\ b^* & c \end{pmatrix}$ where $b = s + it$

The Pauli matrices are well known 2×2
hermitian matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

if we add to these the identity matrix $\mathbb{1}$.
we see we can write

$$\begin{pmatrix} a & s+it \\ s-it & c \end{pmatrix} = \frac{(a+c)}{2}\overset{\mathbb{1}}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} + \frac{(a-c)}{2}\overset{\sigma_3}{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}$$

$$+ \; s \overset{\sigma_1}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} - t \overset{\sigma_2}{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}$$

Thus $\sigma_1 \; \sigma_2 \; \sigma_3 \; \mathbb{1}$ form a complete basis for
2×2 hermitian matrices so all 2×2 hermitian matrices
can be expressed in terms of these.

So $\sigma_i^{\,2} = \mathbb{1}$ and $\mathbb{1}^2 = \mathbb{1}$.

$\{\sigma_i\sigma_j + \sigma_j\sigma_i\} = 0$ since $\sigma_i\sigma_j = i\,\varepsilon_{ijk}\sigma_k$.

BUT $\mathbb{1}\sigma_i + \sigma_i\mathbb{1} \neq 0$

So since the identity matrix commutes and does therefore not anti commute with the rest there are not 4 2×2 anticommuting hermitian matrices (independent).

So the α_i and β cannot be (2×2)

So the next possible dimension is 4×4.

Let $\alpha^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$ $\beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$

{using block matrices}.

i) we can see $\alpha_i \alpha_j + \alpha_j \alpha_i = 0$

$$\alpha_i \alpha_j = \begin{pmatrix} \sigma_i \sigma_j & 0 \\ 0 & \sigma_i \sigma_j \end{pmatrix} \qquad \alpha_j \alpha_i = \begin{pmatrix} \sigma_j \sigma_i & 0 \\ 0 & \sigma_j \sigma_i \end{pmatrix}$$

but since $\sigma_i \sigma_j = i \varepsilon_{ijk} \sigma_k$

$$\alpha_i \alpha_j + \alpha_j \alpha_i = \begin{pmatrix} \sigma_i \sigma_j + \sigma_j \sigma_i & 0 \\ 0 & \sigma_i \sigma_j + \sigma_j \sigma_i \end{pmatrix} = 0.$$

ii) $\alpha_i \beta + \beta \alpha_i = 0.$

$$\alpha_i \beta = \begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix} \qquad \beta \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

$\therefore$ $\alpha_i \beta + \beta \alpha_i = 0$ the β anticommutes with α_i.

iii) clearly

$$\alpha_i^2 = \alpha_i^{+} \alpha_i = \begin{pmatrix} \sigma_i^2 & 0 \\ 0 & \sigma_i^2 \end{pmatrix} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix} = \mathbb{1}.$$

$$\beta^2 = \begin{pmatrix} (\mathbb{1})^2 & 0 \\ 0 & (-\mathbb{1})^2 \end{pmatrix} = \mathbb{1}.$$

The DIRAC EQUATION and SPIN

Take the standard DIRAC equation

$$\hat{E}\psi = (\underline{\alpha}.\underline{p} + \beta m)\psi$$

and set $\underline{p}=0$ so the particle is at rest

$$\therefore \quad \hat{E}\psi = \beta m \psi$$

$$\Rightarrow \quad i\frac{\partial}{\partial t}\psi(t) = m \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \psi(t)$$

For this equation to balance and have the same dimensions on left and right side $\psi(t)$ must be a <u>4- component</u> object

$$\text{Let} \quad \psi(t) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} . e^{-iEt}$$

The first and second equations gives

$$i\frac{\partial\psi}{\partial t} = m\psi(t)$$

$$\Rightarrow \quad Eae^{-iEt} = mae^{-iEt}$$

$$Ebe^{-iEt} = mbe^{-iEt}$$

but at rest the rest mass $m = E$

however the 3rd and 4th equations only work if the energy of the state is negative.

$$3)\quad -Ec\,e^{iEt} = mc\,e^{iEt}$$
$$4)\quad -Ed\,e^{iEt} = mc\,e^{iEt}$$

so in $\psi(x) = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} e^{-iEt}$

a and b components associated with +E states

c d components associated with -E states

Since a and b are associated with the same energy, and we can use

$$\begin{pmatrix}1\\0\\0\\0\end{pmatrix} e^{-iEt} \quad \text{and} \quad \begin{pmatrix}0\\1\\0\\0\end{pmatrix} e^{-iEt} \quad \text{as}$$

two independent basis states of the same energy E we have two degenerate states. Q.M says there must be some hermitian operator which commutes with the hamiltonian which distinguishes between these

[N.B] $\quad \hat{\Theta} = [H, \Theta]$

$$\hat{H} = M\beta = M\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}$$

one such operator is $\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix}$

$$\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix}\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix} = \begin{pmatrix}\sigma_3 & 0\\ 0 & -\sigma_3\end{pmatrix}$$

$$\begin{pmatrix}\mathbb{1} & 0\\ 0 & -\mathbb{1}\end{pmatrix}\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix} = \begin{pmatrix}\sigma & 0\\ 0 & -\sigma_3\end{pmatrix}$$

so $\begin{pmatrix}\sigma^3 & 0\\ 0 & \sigma^3\end{pmatrix}$ can be used to label the degenerate states.