The Electromagnetic Vector Potential

a) $B = \nabla \times A$ since $(\nabla . B = 0)$ and $\nabla . (\nabla \times A) = 0$.

b) $E = -\dot{A} - \nabla\phi$ since $(\nabla \times E = -\dot{B})$ and $\nabla \times \nabla\phi = 0$.

(ϕ is the electrostatic potential.

c) $\nabla \times B = J + \dot{E}$

$\boxed{c = 1}$

d) $\nabla . E = \rho$

① $a + b \to c$ and taking the curl.

$$\nabla \times \nabla \times A = J - \ddot{A} - \nabla\dot{\phi}$$

$$\Rightarrow \quad -\nabla^2 A + \nabla(\nabla . A) = J - \ddot{A} - \nabla\dot{\phi}$$

$$\Rightarrow \quad \boxed{\nabla^2 A - \ddot{A} = -J + \nabla(\nabla . A + \dot{\phi})}$$

② $b \to d$

$$\boxed{-(\nabla . \dot{A}) - \nabla^2\phi = \rho .}$$

But a and b only define $\underline{A}$ and ϕ's gradients and we can the gradient of an arbitrary function χ to A as $\nabla \times \nabla\chi \equiv 0$. Also ϕ can have an arbitrary constant (in x, y, z) added to it and leave E invariant.

Thus.
$$\boxed{\begin{aligned} A &\to A - \nabla\chi \\ \phi &\to \phi + \frac{\partial\chi}{\partial t} \end{aligned}}$$

Summarizing.

$$\left.\begin{array}{lcl} A & \rightarrow & A - \nabla\chi \\ \phi & \rightarrow & \phi + \dfrac{\partial\chi}{\partial t} \end{array}\right\} \chi \text{ arbitrary scalar function.}$$

we can find a χ which enables us to make the following equation true

$$\nabla . A + \dot{\phi} = 0.$$

ie we add a $\chi(x, y, z, t)$ which makes this true everywhere. This is called the "Lorentz condition" or "gauge")

Thus equation ① and ② become.

$$\left.\begin{array}{l} \ddot{\underline{A}} - \nabla^2 \underline{A} = \underline{J} \\ \ddot{\phi} - \nabla^2 \phi = \rho \end{array}\right\} \partial^\mu \partial_\mu A_\mu = j_\mu .$$

where $\underline{A_\mu} = (\phi, \underline{A})$ $\quad \underline{j_\mu} = (\rho, \underline{J})$

Thus

$$\boxed{\Box^2 A^\mu = j^\mu{}_{em}}$$

Thus all we know about electromagnetic theory is expressed in this one 4 vector equation!

So A^μ is the important quantity along with j^μ_{em}.

π^+ scattering from a Coulomb Potential

How to add the electromagnetic potential to the free space equation

we use the "minimal prescription" as for non-relativistic q.m.

$$\hat{\underline{p}} \rightarrow \hat{\underline{p}} - e\,\underline{A} \qquad \hat{E} \rightarrow \hat{E} - e\phi$$

but using $A^\mu = (\phi, \underline{A})$

we can rewrite the above as

$$p^\mu \rightarrow p^\mu - eA^\mu \rightarrow i\partial^\mu - eA^\mu .$$

Thus the Klein - Gordon equation becomes.

$$\Box^2\psi = -\partial^\mu \partial_\mu \psi \rightarrow (i\partial^\mu - eA^\mu)(i\partial_\mu - eA_\mu)\psi = m^2\psi .$$

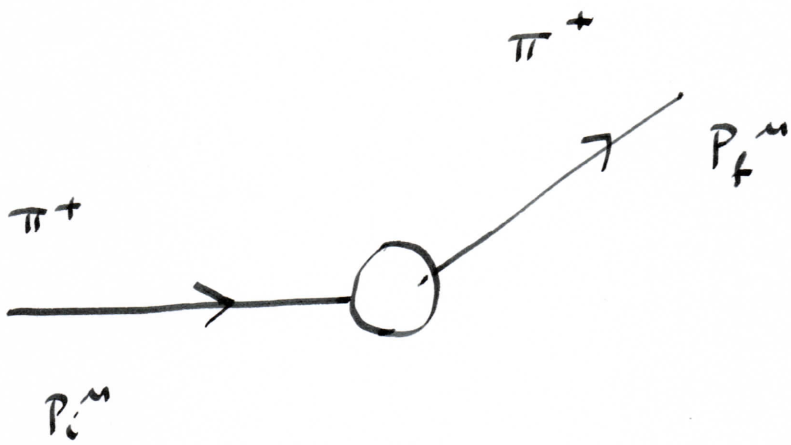
expanding this out and ignoring terms of order e^2.

$$-\partial^\mu\partial_\mu\psi \underbrace{- ie(\partial^\mu A_\mu - A^\mu\partial_\mu)}\psi = m^2\psi .$$

This must be the e.m. perturbing potential

$$\boxed{U(x,t) \equiv ie(\partial^\mu A_\mu - A^\mu \partial_\mu)\psi}$$

So what is the Matrix Element with this potential ?



$$a_f^{(1)} = \int dt \int d^3x \; \psi_f^*(\pi^+, p_f) \; \hat{V}^{(1)} \; \psi(\pi^+, p_i)$$

where $\hat{V}^{(1)} = e(\partial^\mu A_\mu - A^\mu \partial_\mu)$

(1) = first order (e^1 only)

For free particle plane waves

$$\psi_i = \frac{e^{-ip_i^\mu x_\mu}}{(L^3\, 2E_i)^{1/2}}$$

$$\psi_f^* = \frac{e^{ip_f^\mu x_\mu}}{(L^3\, 2E_f)^{1/2}}$$

N.B This normalization preserves the simple single particle interpretation of the K.G. wavefunction

$j^\mu = (\rho, \underline{j})$, $\int \rho \, d^3x = 1$.

$\rho_{K.G.} = \dot{\psi}^* \psi - \psi \dot{\psi}^* = 2E\psi^*\psi$

This part gives $(L^3)^{1/2}$ normalization.

$$\therefore \; a_f^{(1)} = \frac{1}{L^3} \int \frac{e^{ip_f^\mu x_\mu}}{\sqrt{2E_f}} \, ie(\partial_\mu A^\mu + A^\mu \partial_\mu) \, \frac{e^{-ip_i^\mu x_\mu}}{\sqrt{2E_i}} \, d^3x \, dt .$$

This is a 4 dimensional integral.

The right hand term $\sim e^{ip_f^\mu x_\mu} A^\mu \partial_\mu e^{-ip_i^\mu x_\mu} d^3x\,dt$ is simple as the operator simply acts on $e^{-ip_i^\mu x_\mu}$ bringing down a factor of $-ip_i^\mu$. However the left hand term is not so simple:-

$$\int e^{ip_f^\mu x_\mu} \partial_\mu A^\mu e^{-ip_i^\mu x_\mu} d^3x\,dt$$

$$= \int e^{ip_f^\mu x_\mu} \left\{ \frac{\partial}{\partial t}\left(A^0 e^{-ip_i^\mu x_\mu}\right) + \nabla.\left(\underline{A}.e^{ip_i^\mu x_\mu}\right)\right\} d^3x\,dt$$

Considering only the first term (in time)

$$\int_{-\infty}^{\infty} e^{ip_f^\mu x_\mu} \frac{\partial}{\partial t}\left(A^0 e^{-ip_i^\mu x_\mu}\right) d^3x\,dt \qquad \{\text{integrating by parts}\}$$

$$= \Big[e^{ip_f^\mu x_\mu} A^0 e^{-ip_i^\mu x_\mu} \Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} A^0 e^{-ip_i^\mu x_\mu} \frac{\partial}{\partial t} e^{ip_f^\mu x_\mu} dt$$

since $A^0(\pm\infty) = 0$.

$$= -ip_f^0 \int e^{ip_f^\mu x_\mu} A^0 e^{-ip_i x_\mu} d^3x\,dt$$

similarly with the $\underline{A}$ term we get

$$i\underline{p_f} \int e^{ip_f^\mu x_\mu} \underline{A}\, e^{-ip_i^\mu x_\mu} d^3x\,dt$$

Thus

$$a_f^{(1)} = \frac{1}{L^3}\int \frac{e^{ip_f^\mu x_\mu}}{\sqrt{2E_f}}\, ie(-ip_f^\mu - ip_i^\mu)\, A_\mu(x)\, \frac{e^{-ip_i^\mu x_\mu}}{\sqrt{2E_i}}\, d^3x\, dt$$

however the potential in all scattering theories is considered to switch on and off for only a short period of time during which it is essentially independent of time.

Thus the time part of the integral can be written as

$$\int dt\; e^{ip_f^0 t - ip_i^0 t} = -i\, 2\pi\, \delta(p_f^0 - p_i^0)$$

$$\equiv -i\, 2\pi\, \delta(E_f - E_i)$$

Thus we can allway write the amplitude as

$$a_f^{(1)} = -i2\pi\, \delta(E_f - E_i)\; V_{fi}$$

where

$$V_{fi} = \int \frac{1}{L^3}\, \frac{e^{-i(p_i - p_f)^\mu x_\mu}}{2E}\, e(p_f + p_i)_\mu A^\mu(x)\, d^3x$$

$$\therefore \quad \boxed{V_{fi} = \frac{e(P_f + P_i)_\mu}{L^3 2E} A^\mu(q)}$$

$$\text{where} \quad A^\mu(q) = \overbrace{\int d^3x\, e^{-i\underline{q}.\underline{x}} A^\mu(x)}^{\text{fourier transform.}}$$

where $\underline{q} = \underline{P}_f - \underline{P}_i$ momentum transfer.

Another form for $A^\mu(q)$

we showed $\Box^2 A^\mu(x) = j^\mu{}_{em}$.

taking the Fourier transform of both sides

$$\int e^{i\underline{q}.\underline{x}} \Box^2 A^\mu(x)\, d^3x = \int e^{i\underline{q}.\underline{x}} j^\mu_{em}(x)\, d^3x .$$

integrating the l.h.s. by parts

$$\Big[e^{i\underline{q}.\underline{x}} \partial_\mu A^\mu(x) \Big]_{-\infty}^{\infty} - iq \int_{-\infty}^{\infty} e^{iq.x} \partial_\mu A^\mu(x)\, d^3x$$

and again

$$- \Big[iq\, e^{+i\underline{q}\underline{x}} A^\mu(x) \Big]_{-\infty}^{\infty} + (iq)^2 \int_{-\infty}^{\infty} A^\mu(x)\, e^{i\underline{q}.\underline{x}}\, d^3x .$$

$$\therefore \quad \boxed{-q^2 A^\mu(q) = j^\mu_{em}(q)} \Rightarrow \boxed{A^\mu(q) = -\frac{j^\mu_{em}(q)}{q^2}}$$

Thus we can write

$$V_{fi} = (p_f + p_i)_\mu \left(-\frac{1}{q^2}\right) j^\mu_{em}(q)$$

Specific Example of Coulomb Scattering:

$$A^\mu(x) = (Ze/4\pi|x|,\ 0)$$

Taking the F.T.

$$A^\mu(q) = \left(\frac{Ze}{q^2},\ 0\right)$$

$$\therefore \quad V_{fi} = \frac{e\ (p_f + p_i)^\mu}{L^3\, 2E} A_\mu(q)$$

$$\therefore \quad \boxed{V_{fi} = \frac{Ze^2}{L^3\, q^2}}$$

The current interpretation

Lets look at this matrix element calculation again:

$$a_f^{(1)} = -i 2\pi \, \delta(E_f - E_i) \, V_{fi}$$

$$V_{fi} = ie \int \psi_f^* \, (\partial_\mu A^\mu + A^\mu \partial_\mu) \, \psi_i \, d^3x \, .$$

this time do not put in an explicit form for ψ_i, ψ_f.

rewriting this term.

$$ie \int \psi_f^* \, \partial_\mu A^\mu \psi \, d^3x = \Big[\psi_f^* A^\mu \psi_i \Big]_{-\infty}^{\infty} - \int \partial_\mu \psi_f^* \, A^\mu \psi_i \, d^3x \, .$$

$$\Big[\psi_f^* A^\mu \psi_i \Big]_{-\infty}^{\infty} = 0$$

(as $A(\pm\infty) = 0$)

$$\therefore \; V_{fi} = +ie \int \left(\psi_f^* \partial_\mu \psi_i - \psi_i \partial_\mu \psi_f^* \right) A^\mu \, d^3x$$

but we can write

$$j_\mu^{fi}(x) = ie \left(\psi_f^* \partial_\mu \psi_i - \psi_i \partial_\mu \psi_f^* \right)$$ - 4 vector current.

$$\boxed{V_{fi} = \int j^\mu_{fi}(x) \, A_\mu(x) \, d^3x}$$

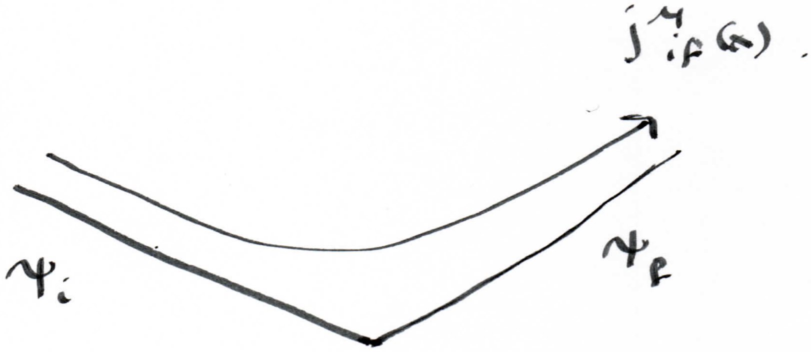
$$\equiv \underbrace{\int \rho \phi \, d^3x}_{\text{electrostatic}} - \underbrace{\int \underline{j} \cdot \underline{A} \, d^3x}_{\text{Magnetic}}$$

What is the meaning of $j^{\mu}{}_{if}(x)$:

$\langle \pi^+ f |\ j^{\mu}(x)\ | \pi^+ i \rangle$ plane wave solution

$$= \frac{e\,(p_i + p_f)_{\mu}\ e^{i(p_f - p_i)^{\mu} x_{\mu}}}{\sqrt{L^3\, 2E_f\, 2E_i}}$$

This is there particle (or em) current between the initial and final states of the scattering particle.



Conserved Current

if $\partial_{\mu} \langle \pi^+ f | j^{\mu}(x) | \pi^+, i \rangle = 0$ — show this!

remember that for $j^{\mu}{}_{em} = (\rho, \underline{J})$

$\partial^{\mu} j_{\mu} = 0.$

$\therefore\ \dfrac{\partial \rho}{\partial t} = \nabla . J$

$\Rightarrow\ \dfrac{\partial}{\partial t} \int d^3x\, \rho = \int \nabla . J\ d^3x = \int \underline{J} . d\underline{S} = 0.$

$\therefore\ \boxed{\dfrac{\partial Q}{\partial t} = 0}$ conserved charge. all infinity.

$$\int \langle \pi^+ f | \hat{j}_0(x) | \pi^+ i \rangle \, d^3x = \int \langle \pi^+ f | (\psi_f^* \partial_\mu \psi - \psi_i \partial_\mu \psi_f^*) | \pi^+ i \rangle \, d^3x$$

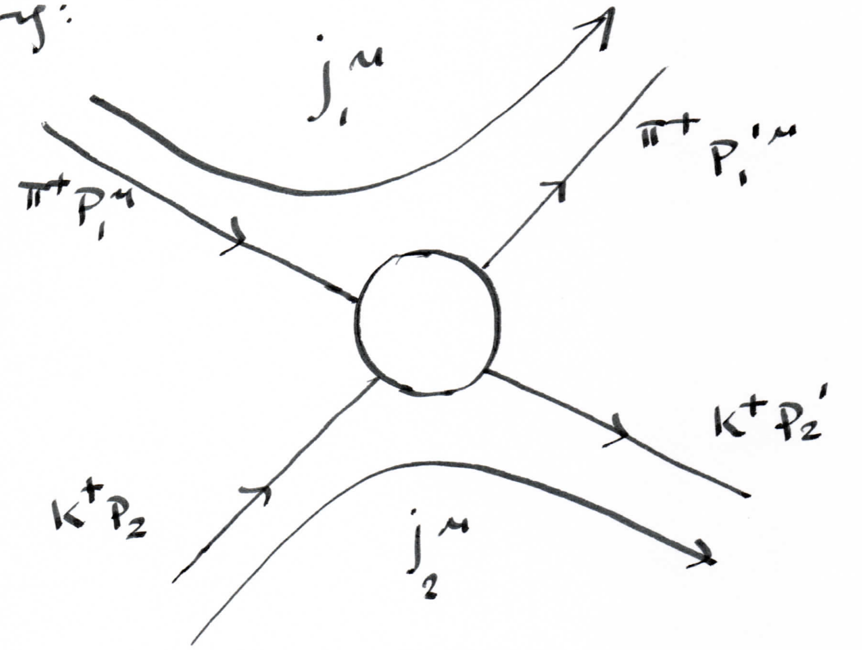
$$= \frac{e}{\sqrt{2E_f E_i}} (E_i + E_f) \frac{1}{L^3} \underbrace{\int e^{i(E_f - E_i)x^0 - (\underline{p}_f - \underline{p}_i) \cdot \underline{x}} \, d^3x \, dt}_{\delta(E_f - E_i \, , \, p_f - p_i)}$$

$$= e.$$

Thus the zeroth term of $j^\mu_{if}(x)$ just gives the conservation of charge, as you would expect.

$\pi^+ K^+ \rightarrow \pi^+ K^+$

Thus far we have only looked at the current interaction off a static coulomb potential. Supposing we want to calculate $\pi^+ K^+ \rightarrow \pi^+ K^+$ scattering:



Assuming we have plane wave solutions: $\psi_1 = A e^{-i p_1^\mu x_\mu}$

$$j^1_\mu(x) = ie\left\{ \psi_{1'}^* \partial_\mu \psi_1 - \psi_1 \partial_\mu \psi_{1'}^* \right\}$$

$$= \frac{e\,(p_1 + p_{1'})_\mu \; e^{i(p_{1'} - p_1)^\mu x_\mu}}{(L^3 2E_1)^{1/2}\,(L^3 2E_{1'})^{1/2}}$$

$$j^2_\mu(x) = ie\left\{ \psi_{2'}^* \partial_\mu \psi_2 - \psi_2 \partial_\mu \psi_{2'}^* \right\}$$

$$= \frac{e\,(p_2 + p_{2'})^\mu \; e^{i(p_{2'} - p_2)^\mu x_\mu}}{(L^3 2E_2)^{1/2}\,(L^3 2E_{2'})^{1/2}}$$

Let $\quad j^{2\,em}_{\mu}(x) \;=\; \Box^2 A_\mu(x)$

$$A_\mu(x) = \frac{-1}{(P_2'-P_2)^2}\, j^2_\mu(x)$$

So substituting this in the amplitude

$$a_f^{(1)}(\pi^+ K^+) = -i\int \frac{1}{(L^3 2E_1')^{1/2}}\, e^{iP_1'^\mu x_\mu}\, ie\left(\partial_\mu A^\mu - A^\mu \partial_\mu\right) \frac{e^{-iP_1^\mu x_\mu}}{(L^3 2E_1)^{1/2}}\, d^4x$$

$$= -i\,\frac{-1}{(P_2'-P_2)^2}\cdot\frac{1}{L^{12}\,(2E_1'\,2E_1\,2E_2'\,2E_2)^{1/2}}$$

$$\cdot\; ie^2\int e^{iP_1'^\mu x_\mu}\left\{\partial_\mu (P_2'+P_2)^\mu e^{i(P_2'-P_2)^\mu x_\mu}\right.$$

(iP_1')

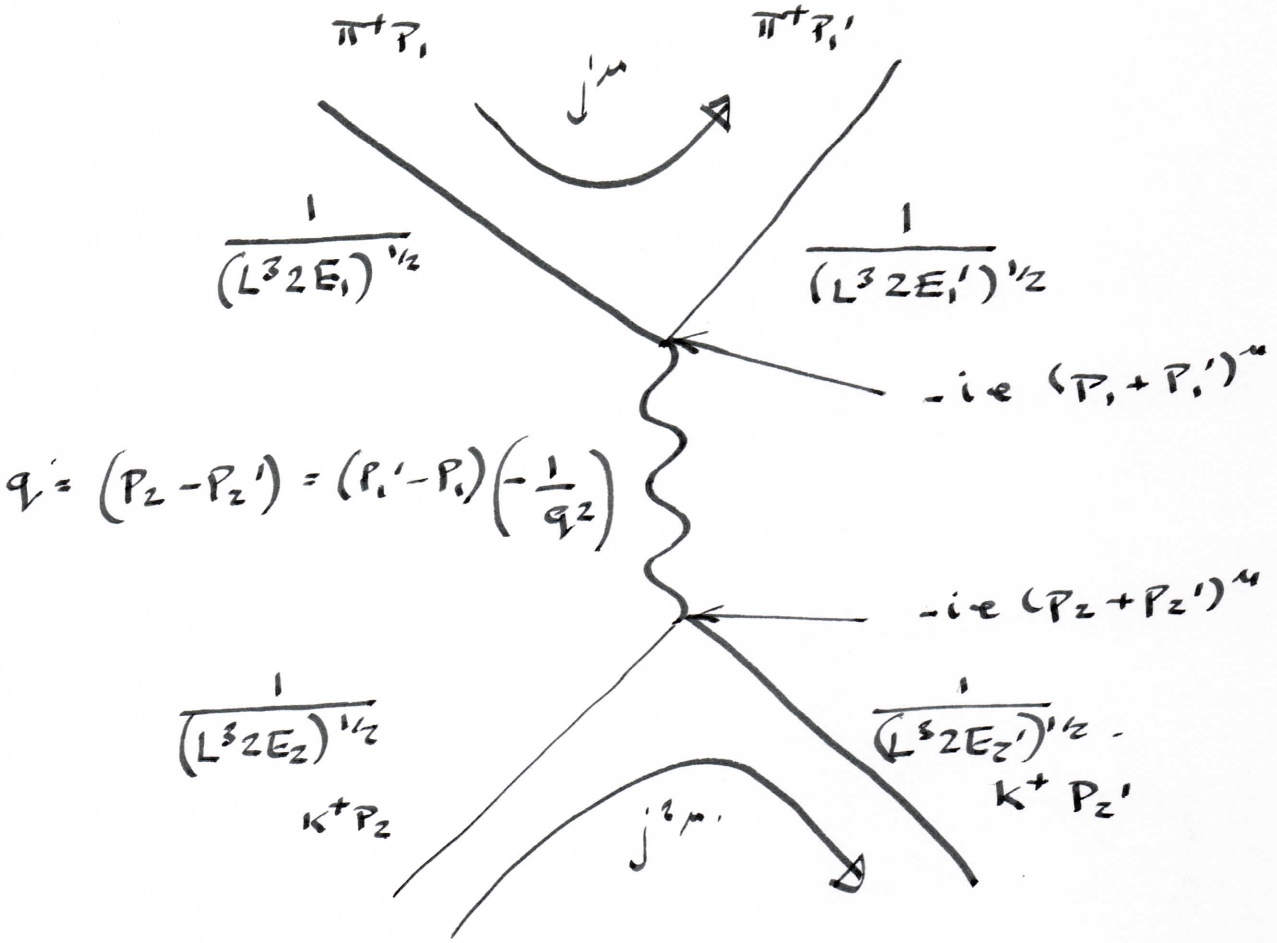
$$\left. + (P_2'+P_2)^\mu e^{i(P_2'-P_2)^\mu x_\mu}\,\partial_\mu\right\} e^{iP_1^\mu x_1}\, d^4x.$$

$\hookrightarrow \; iP_1^\mu$.

$\therefore$

$$\boxed{a_f^{(1)}(\pi^+ K^+) = -i\,\frac{-1}{(P_2'-P_2)^2}\,\frac{1}{(16\,L^{12}\,E_1 E_1' E_2 E_2')^{1/2}}\; e^2\,(P_1'+P_1)_\mu\,(P_2'+P_2)^\mu\;(2\pi)^4\,\delta(\hat{P_1'}+\hat{P_2'}-\hat{P_1}-\hat{P_2})}$$

Feynman Diagram for spin 0 scattering:

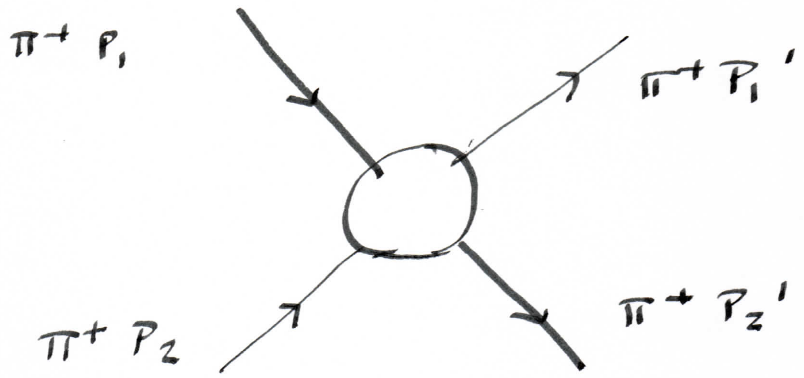


$$= \frac{j^1_\mu(x)\; J^{\mu 2}(x)}{q^2}$$

Thus the scattering can be thought of as just two electromagnetic currents interacting as above.

$$a_{fi}(\pi^+K^+) = \frac{i(2\pi)^4\,\delta^4(p_1'+p_2'-p_1-p_2)}{L^{12}\,16E_1E_1'E_2E_2')^{1/2}} \cdot \left\{ -e^2\,\frac{(p_1+p_1')_\mu\,(p_2+p_2')^\mu}{(p_2-p_2')^2} \right\}$$

This part is called the invariant amplitude



Now we have identical bosons we must have a symmetric wave function (by exchanging p_1 and p_2).

In general if we have a function which we want to make symmetric under interchange of 2 variables. eg $f(x,y)$ under interchange of x and y.

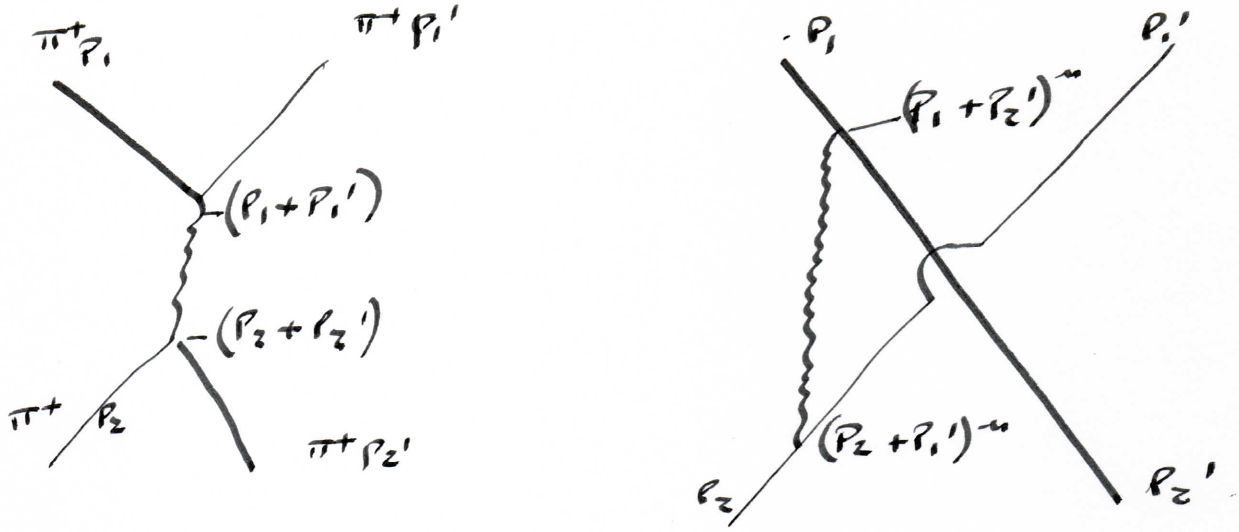
$$f(x,y) + f(y,x)$$ must be symmetric under $x \to y$.

Thus writting down the symmetric wave function gives.

$$a_f^{(1)}(\pi^+\pi^+) = \frac{i(2\pi)^2\delta^4(p_1'+p_2'-p_1-p_2)}{(L^{12}\,16\,E_1E_{1'}E_2E_{2'})^{1/2}} \cdot$$

$$\left\{ -e^2\frac{(p_1+p_1')_\mu(p_2+p_{2'})^\mu}{(p_2-p_{2'})^2} - e^2\frac{(p_1+p_2')_\mu(p_2+p_1')^\mu}{(p_2-p_1')^2} \right\}$$

This is symmetric under $p_{1'} \rightleftarrows p_2'$ also $p_1 \rightleftarrows p_2$.



(t - channel) (u-channel)

So $$a_f^{(1)} = \frac{i(2\pi)^4 \delta^4(P_1'+P_2'-P_1-P_2)}{(L^{12}16E_1E_1'E_2E_2')^{1/2}} F_{\pi^+\pi^+\to\pi^+\pi^+}$$

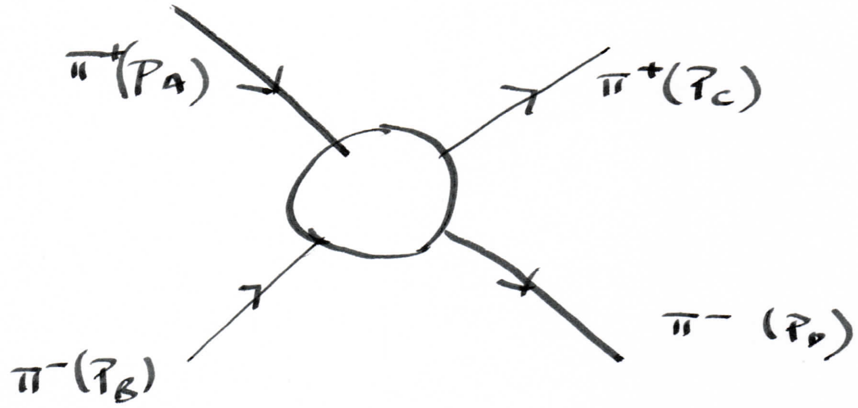
where

$$F_{\pi^+\pi^+\to\pi^+\pi^+} = -e^2\frac{(P_1+P_1')_\mu (P_2+P_2')^\mu}{(P_2-P_2')^2}$$

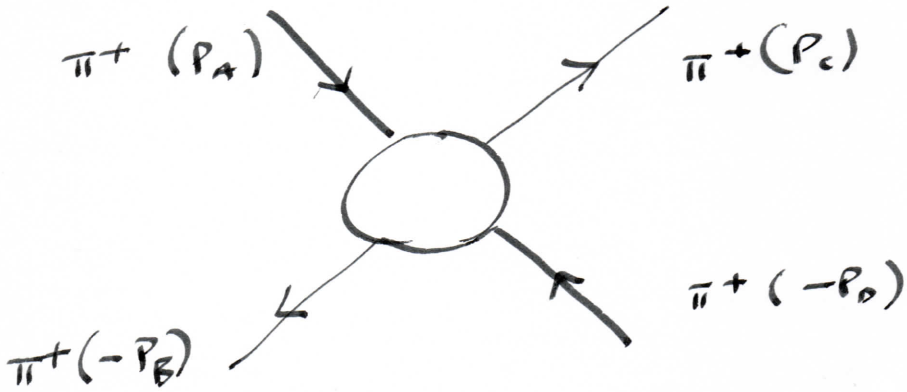
$$-e^2\frac{(P_1+P_2')_\mu (P_2+P_1')^\mu}{(P_2-P_1')^2}.$$

$\pi^+\pi^- \to \pi^+\pi^-$ Scattering

$$\pi^+(p_A) + \pi^-(p_B) \to \pi^+(p_C) + \pi^-(p_D)$$



"crossing symmetry" (making a π^- from a π^+) says this amplitude is the same as.

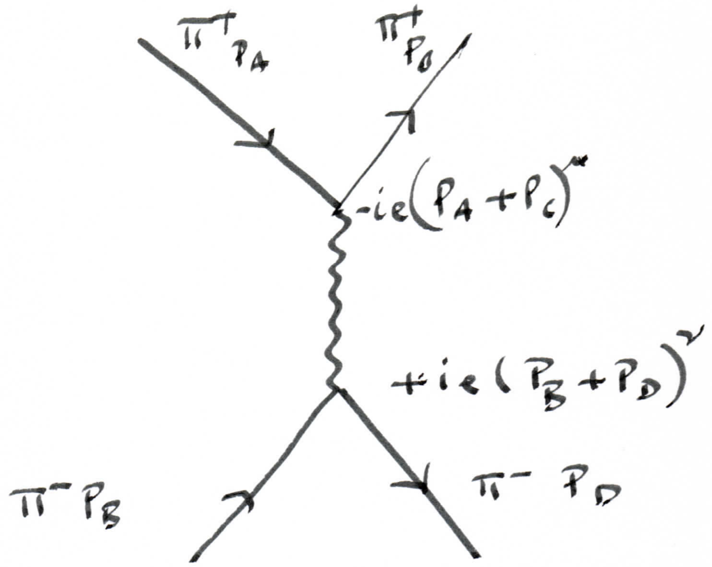


$$\therefore F_{\pi^+\pi^- \to \pi^+\pi^-}(p_A, p_B; p_C, p_D) = F_{\pi^+\pi^+ \to \pi^+\pi^+}(p_A, -p_D; p_C, -p_B)$$

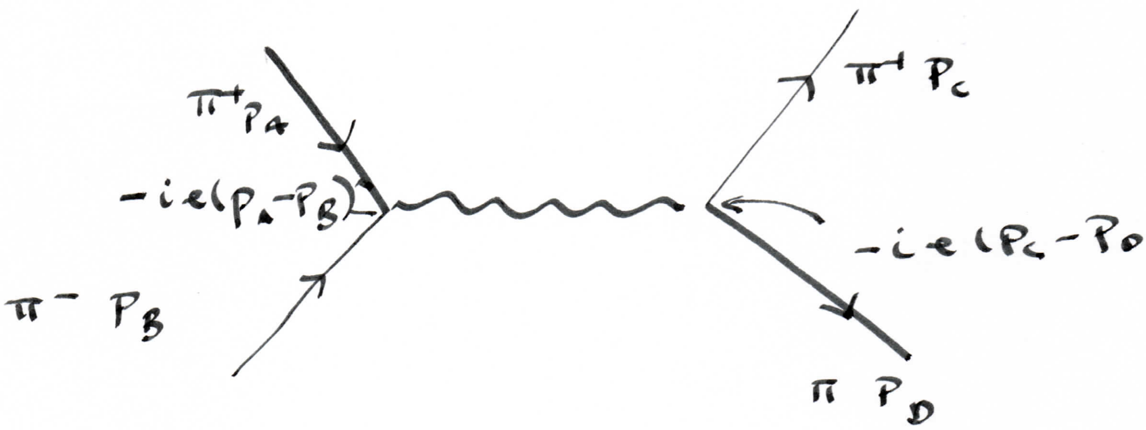
Thus.

$$F_{\pi^+\pi^- \to \pi^+\pi^-} \qquad \boxed{p_C \longleftrightarrow -p_B}$$

$$= \left\{ -e^2 \frac{(p_A + p_C)_\mu (-p_D - p_B)^\mu}{(-p_B + p_D)^2} - e^2 \frac{(p_A - p_B)_\mu (-p_D + p_C)^\mu}{(p_C + p_D)^2} \right\}$$

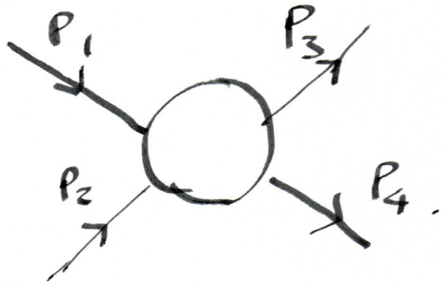


t - channel.



s - channel.

Mandlestram Variables



$$s = (p_1 + p_2)^2$$

$$t = (p_1 - p_3)^2 = (p_2 - p_4)^2$$

$$u = (p - p_4)^2 = (p_2 - p_3)^2$$

These invariants represent the exchange 4-momentum in each of the channel.

The invariant amplitudes can be rewritten in terms of these invariants:

eg.

$$F_{\pi^+\pi^+ \to \pi^+\pi^-} = e^2 \left\{ \frac{s-u}{t} + \frac{t-u}{s} \right\} .$$

Worksheet 3.

Question 1

Consider the Invariant Amplitude F for electromagnetic $\pi^+\pi^-$ scattering to order e^2

a) In the centre of mass system of the $\pi^+\pi^-$ let the energy of either particle be E, the magnitudes of either of its velocity and momentum be v and $|\underline{p}|$ and the C.M.S scattering angle be θ.

Show that

$$F = e^2\left[\frac{E^2}{p^2}\left\{\frac{1 + v^2\cos^2(\theta/2)}{\sin^2(\theta/2)}\right\} - v^2\cos\theta\right]$$

b) What is the behaviour of F near $\theta = 0$.

How would this change if the photon had a finite rest mass.