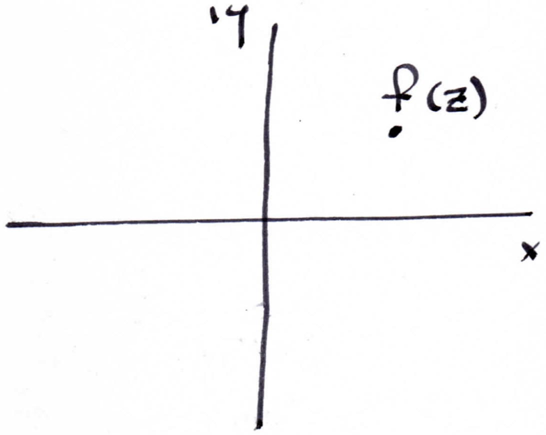
Functions of a complex Variable

It is a simple extension to define a function of a complex number z.



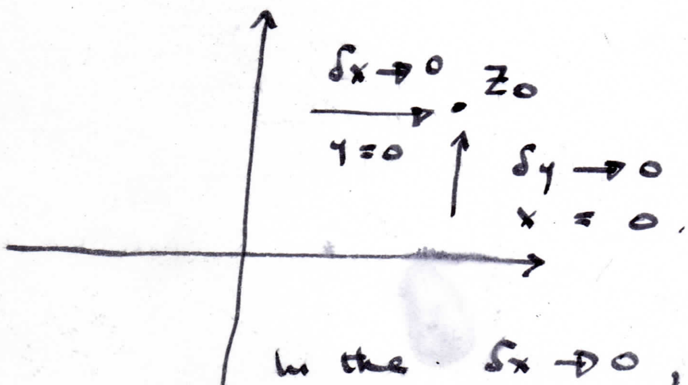
$z = x + iy.$

Cauchy-Riemann Conditions

If the function is to differentiable at any given point in the complex plane what conditions would make that the same value irrespective of the direction from which the point is approached.

$$\delta z = \delta x + i\delta y$$

$$\delta f = \delta u + i\delta v$$



$$\frac{\delta f}{\delta z} = \frac{\delta u + i\delta v}{\delta x + i\delta y}$$

In the $\delta x \to 0$, $y = 0$ direction

$$\left.\frac{\delta f}{\delta z}\right|_{\delta x \to 0} = \lim_{\delta x \to 0}\left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right)$$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

In the $\delta y \to 0$ $x = 0$ direction

$$\frac{\delta f}{\partial z} = \lim_{\delta y \to 0} \left(\frac{\delta u}{i\,\delta y} + \frac{i\,\delta v}{i\,\delta y} \right)$$

$$= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

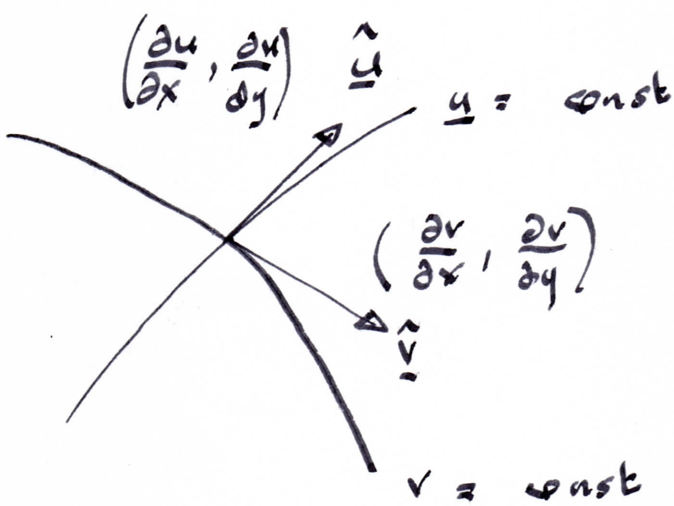
If these two are to be equivalent the real and imaginary part must be equal.

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}}$$

These are known as the Cauchy-Riemann Conditions.

N.B. if a function $f(z)$ obeys these conditions

then $\nabla^2 u = \nabla^2 v = 0$ (both obey Laplace's eqn)



$$\hat{u} \cdot \hat{v} =$$

$$\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}$$

$$\frac{\partial v}{\partial y}\frac{\partial v}{\partial x} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = 0.$$

Thus u = const v = const are everywhere orthogonal.

Stokes Theorem

$$\oint \underline{A} \cdot d\underline{L} = \int \underline{\nabla} \times \underline{A} \cdot \underline{ds}$$

The area enclose by path.

$$\oint f(z) \cdot dz$$

[f(z) and dz are vectors on the complex plane]

$$\oint (u + iv)(dx + i\,dy)$$

$$= \oint (u\,dx - v\,dy) + i(v\,dx + u\,dy)$$

Now since

$$\nabla \times (u, -v) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ u & -v \end{vmatrix} = -\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\nabla \times (v\ u) = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ v & u \end{vmatrix} = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)$$

$$= \int -\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\,dy + i \int \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

$= 0$ from the Cauchy Riemann conditions

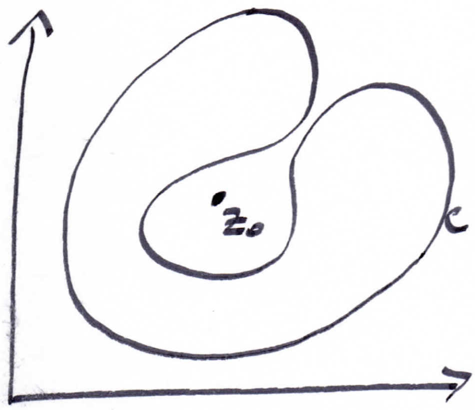
Thus

$$\boxed{\oint f(z)\, dz = 0. \qquad \text{Cauchy Theorem.}}$$

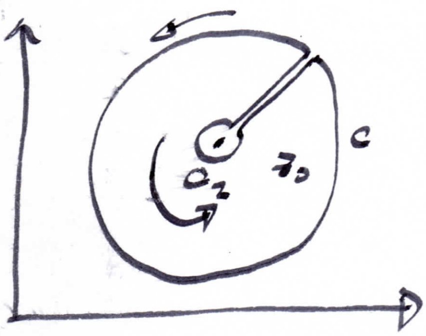
Lets consider the function

$$\frac{f(z)}{z - z_0}$$

This function is assumed analytic everywhere except at $z = z_0$.



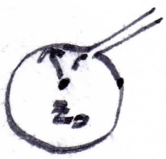
$$\oint_C f(z)\, dz = 0.$$



$$\oint_{C + C_2} f(z)\, dz = 0$$

$$\therefore \oint_C \frac{f(z)}{z - z_0}\, d = \oint_{C_2} \frac{f(z)}{z - z_0}\, dz.$$

But if we let $z = z_0 + re^{i\theta}$



$$[dz = ie^{i\theta} d\theta]$$

$$= \oint_{C_2} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}}\, ie^{i\theta}\, d\theta$$

$$\therefore \lim r \to 0 \qquad = i f(z_0) \int_0^{2\pi} d\theta$$

$$\boxed{\oint_C \frac{f(z)}{(z - z_0)}\, dz = 2\pi i\, f(z_0)}$$

Theorem of residues.

The derivative of $f(z)$

$$f(z') = \lim_{\delta z \to 0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{1}{2\pi i\,\delta z} \left(\oint \frac{f(z)}{z - z_0 - \delta z} dz - \oint \frac{f(z)}{z - z_0} dz \right)$$

$$= \lim_{\delta z \to 0} \frac{1}{2\pi i\,\delta z} \left(\oint \frac{\delta z\, f(z)}{(z - z_0 - \delta z)(z - z_0)} \right) dz$$

$$= \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^2} dz .$$

Or for higher orders.

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint \frac{f(z)\, dz}{(z - z_0)^{n+1}}$$

$$\boxed{\oint \frac{f(z)}{(z - z_0)^{n+1}} = \frac{2\pi i}{n!} \left. \frac{d^n}{dz^n} f(z_0) \right|_{z_0} .}$$

Relativistic Q.M.

Course work

Determine the nature of the singularities of each of the following functions and evaluate the residues.

(1)

i) $\dfrac{1}{z^2+a^2}$

ii) $\dfrac{1}{(z^2+a^2)^2}$

iii) $\dfrac{z^2}{(z^2+a^2)^2}$

iv) $\dfrac{\sin 1/z}{(z^2+a^2)}$

v) $\dfrac{z e^{iz}}{z^2+a^2}$

vi) $\dfrac{z e^{+iz}}{(z^2-a^2)}$

(2) Prove that

$$\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$$

hint (or otherwise)

$$\sin^2 x = \frac{1}{2}(1-\cos 2x)$$

RELATIVISTIC QUANTUM MECHANICS

Definitions:

4-vector

covariant 4-vector $\quad x_\mu = (x_0, x_1, x_2, x_3)$

contravariant 4-vector $\quad x^\mu = (x^0, x^1, x^2, x^3)$

A 4-vector transforms under a Lorentz Boost (in the x direction) as

$$\begin{pmatrix} x_0' \\ x_1' \\ x_2' \\ x_3 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

4 vectors are very useful when formulation a relativistically invariant system.

Metric Tensor

Defined to be

$$g^{\nu\mu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The contra and covariant 4-vectors are related as follows.

Summation over the repeated index is called "contraction"

$$x^{\nu} = g^{\nu\mu} x_{\mu} \quad (g^{\nu\mu}x_{\mu} = g^{\nu 0}x_0 + g^{\nu 1}x_1 + g^{\nu 2}x_2 + g^{\nu 3}x_3)$$

Thus

$$(x^0, x^1, x^2, x^3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Thus

$$x^{\mu} x_{\mu} = g^{\mu\nu} x_{\nu} x_{\mu}$$

$$= x_0^2 - x_1^2 - x_2^2 - x_3^2$$

Thus if we are dealing with $(E, P_x, P_y\ P_z) = x_{\mu}$
or $(t, x, y, z) = x_{\mu}$.

$$= E^2 - P_x^2 - P_y^2 - P_z^2$$

$$\text{or } t^2 - x^2 - y^2 - z^2$$

Both of which are relativistically INVARIANT quantities.

The "dot product" defined a "distance squared" in a given space. In ordinary 3-space under rotation

$$x_\mu x_\mu = x^2 + y^2 + z^2$$ which is invariant under rotation

In Lorentz space we define

$$g^{\nu\mu} x_\mu x_\nu = x_0^2 - x_1^2 - x_2^2 - x_3^2$$ which is the invariant under a Lorentz Boost.

we could have defined a $g^{\nu\mu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

for the first case and in general for a given space x_μ we define a distance as

$$g^{\nu\mu} x_\mu x_\nu$$ to give us invariant "distance".

This is and relativistically invariant formulation terms like $g^{\nu\mu} x_\mu x_\nu \equiv x^\nu x_\nu$ will frequently occur.

KLEIN GORDON EQUATION

The relativistic energy-momentum equation is $(c=1)$

$$E^2 = p^2 + m^2$$

If we make the same operator substitutions

$$\hat{E}^2 \psi = \hat{p}^2 \psi + m^2 \psi .$$

where $\hat{E} = i \dfrac{\partial}{\partial t}$ $\qquad \underline{\hat{P}} = -i \underline{\nabla}$

$$- \frac{\partial^2}{\partial t^2} \psi = \left(-\nabla^2 + m^2\right) \psi$$

If we define the D'Alembertian Operator

$$\Box^2 \equiv \left(\frac{\partial^2}{\partial t^2} - \nabla^2 \right)$$

We can rewrite the above equation as.

$$\left(\Box^2 + m^2\right) \psi = 0 .$$

This can be written in a very neat form if we use 4-vectors.

If $p^\mu = (E, \underline{p})$

we can write $\hat{p}^\mu = i\partial^\mu$

where $\partial^\mu = \left(\frac{\partial}{\partial t}, \underline{\nabla}\right)$

Thus we can write the KGE as

$$\boxed{\begin{array}{l} p^\mu p_\mu \psi = m^2 \psi \\ = \partial^\mu \partial_\mu \psi = m^2 \psi \\ \Box^2 = \partial^\mu \partial_\mu . \end{array}}$$

Solutions to the K.G.E.

$$\text{if} \quad \psi = A e^{-ip^\mu x_\mu} \equiv A^{-i\omega t + i\underline{k}\cdot\underline{\omega}}$$

if we substitute a solution of this form into the K.G.E. we find

$$\omega^2 = p^2 + m^2$$

$$\omega = \pm\sqrt{p^2 + m^2}$$

Thus $\pm$ energy are possible !

The current interpretation of the wave equation :-

Schroedinger:

$$① \quad \left(i\frac{\partial}{\partial t} - \frac{\nabla^2}{2m} + m^2\right)\psi = 0$$

Complex Conjugate

$$② \quad \left(-i\frac{\partial}{\partial t} - \frac{\nabla^2}{2m} + m^2\right)\psi^* = 0$$

ψ^* ① $-$ ψ ②

$$i\left(\psi^*\frac{\partial}{\partial t}\psi + \psi\frac{\partial}{\partial t}\psi^*\right) - \left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) = 0$$

$$= i\,\frac{\partial\psi^*\psi}{\partial t} - \nabla.\left(\psi^*\underline{\nabla}\psi - \psi\underline{\nabla}\psi^*\right) = 0.$$

Now if we remember Stokes or Gauss' Theorem

$$\int J.\,ds = \int \nabla.J\,dv$$

Surface integral — Volume integral.

We can integrate the last equation over all of space.

$$\int i \frac{\partial \psi^* \psi}{\partial t} \, dv = \int \nabla . (\psi^* \underline{\nabla} \psi - \psi \nabla \psi^*) \, dv$$

$$= \int (\psi^* \underline{\nabla} \psi - \psi \underline{\nabla} \psi^*) . \underline{ds}$$

$$= \frac{i \partial \int \psi^* \psi \, dv}{\partial t} = \int (\psi^* \nabla \psi - \psi \nabla \psi^*) . ds .$$

This is a flux equation. The rate of change of the quantity on the left hand side ($\int \psi^* \psi \, dv$) is equal to the flux quantity flowing out of the surface enclosing the volume. $(\psi^* \nabla \psi - \psi \nabla \psi^*)$. [N.B. $\underline{ds}$ is the vector area and it therefore $\perp$ to the area surface. Thus $(\psi^* \underline{\nabla} \psi - \psi \underline{\nabla} \psi^*) . \underline{ds}$ represent the normal flux flowing out of the volume]

Thus

$\int \psi^* \psi \, dv$ is a conserved quantity (in time) if there is no flow out of the volume, and if the volume is all of space then there cannot be.

$(\psi^* \nabla \psi - \psi \nabla \psi^*)$, is a flux.

On the strength of these observations

$\psi^*(x)\psi(x)$ was associated with the probability of a particle at a given point (x)

$(\psi^* \nabla \psi - \psi \nabla \psi)$ was associated with the flux or flow of a particle probability.

Thus if a flux is to be continuous between two regions 1, 2,

$$\psi_1^* \nabla \psi_1 - \psi_1 \nabla \psi_1 \;\Big|\; \psi_2^* \nabla \psi_2 - \psi_2 \nabla \psi_2^*$$

$\therefore$

$$\left.\begin{aligned} \psi_1 &= \psi_2 \\ \nabla \psi_1 &= \nabla \psi_2 \end{aligned}\right\}$$ Are the necessary and sufficient boundary conditions.

Klein - Gordon Equation

$$① \quad \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) \psi = 0 .$$

Complex conjugate.

$$② \quad \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) \psi^* = 0 .$$

$\therefore \psi^* ① - \psi ②$

$$= \psi^* \frac{\partial^2 \psi}{\partial t^2} - \psi \frac{\partial^2 \psi^*}{\partial t^2} - \left(\psi^* \nabla^2 \psi - \psi \nabla^2 \psi^* \right) = 0$$

$$= \frac{\partial}{\partial t} \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) - \nabla . \left(\psi^* \nabla \psi - \psi \nabla \psi^* \right) = 0 .$$

N.B. for both these terms the cross terms generated in each cancel.

Thus $\displaystyle\int \frac{\partial}{\partial t} \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) dv = \int \underline{\nabla} . \left(\psi^* \underline{\nabla} \psi - \psi \underline{\nabla} \psi^* \right) dV$

volume integral

using gauss' theorem as before.

$$\Rightarrow \frac{\partial}{\partial t} \int \left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} \right) dv = \int \psi^* \underline{\nabla} \psi - \psi \nabla \psi^* . \underline{dS}$$

Surface integral

So once more as for Schroedinger we have a "flux" equation. The right hand side is the flux flowing through a surface and the left hand side is a quanity which is changing with time according to the flow of "flux".

So now the probability is associated with the term.

$$\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t} = \psi^* \dot{\psi} - \psi \dot{\psi}^*$$

The flux is however the same as for Schroedinger

$$J = (\psi^* \underline{\nabla} \psi - \psi \underline{\nabla} \psi^*).$$

<u>The current (kg)</u>

if we write $j^\mu = ((\psi^* \dot{\psi} - \psi \dot{\psi}^*), \psi^* \underline{\nabla} \psi - \psi \underline{\nabla} \psi^*)$

Then the flux equation can be written as.

$$\partial^\mu j_\mu = \frac{\partial}{\partial t} (\psi^* \dot{\psi} - \psi^* \psi^*) - \nabla \cdot (\psi^* \underline{\nabla} \psi - \psi \nabla \psi^*) = 0.$$

where $\partial^\mu \equiv \left(\frac{\partial}{\partial t}, \underline{\nabla}\right)$

The K.G.E is relativistically invariant and the physics which is represented by the flux equation is too so it is not surprising that we can write the formula in this form.

$$\partial^{\mu} j_{\mu} = 0.$$

Note since we know ∂^{μ} is a 4-vector so is j^{μ}.

Interpretation of the 4-vector j^{μ}.

If we plug in the plane wave solution to the current wave functions

$$\psi = A e^{i p^{\mu} x_{\mu}} = A e^{i(\underline{k}.\underline{x} - Et)}$$

$$\psi^{*}\dot{\psi} - \psi\dot{\psi}^{*} = iEA$$

$$\psi^{*}\nabla\psi - \psi\nabla\psi^{*} = i\underline{p}A$$

$$\therefore \quad j^{\mu} = 2A(iE, i\underline{p})$$

Thus we can immediately see j^{μ} is a 4-vector.

define $j^{\mu'} = \frac{1}{i} j^{\mu} = 2A(E, \underline{p}).$

What about the probabilistic interpretation:

Schroedinger $\quad j^0 = \psi^*\psi$ — interpreted as the probability.

$$\int j^0 \, dv = \int_{-\infty}^{\infty} \psi^*\psi \, dv = 1$$

since the wave function represents one particle somewhere in space.

K.G. $\quad j^0 = \psi^*\dot{\psi} - \psi\dot{\psi}^*$

$$\therefore \int j^0 \, dv = \int \psi^*\dot{\psi} - \psi\dot{\psi}^* \, dv = 2E$$

number of particles.

Thus to recover the probabilistic interpretation in terms of the number of particles

$$\psi_N^{KG} = \frac{1}{\sqrt{2E}} \psi^{KG}$$

Thus for normalized KG.

$$j_N^{KG^\mu} = \left(1, \frac{p}{2E}\right)$$

But this is no longer manifestly covariant!

Interpretation of the negative energy solutions

We have established that the K. G. particles can be expressed as a 4-vector current

$$j^\mu = (\psi^*\dot\psi - \psi\dot\psi^*, \psi^*\underline{\nabla}\psi - \psi\underline{\nabla}\psi^*)$$

where $\partial^\mu j_\mu = 0$.

If this 4-vector can be thought of as a particle flux or particle current then perhaps this current can be related to the electromagnetic 4-current as follows:

$$\boxed{e j^\mu \equiv j^\mu_{em}.}$$

For a free particle (K.G.) $\psi = A e^{i p^\mu x_\mu}$.

$p^\mu = (E, \underline{p})$ where $E = \pm\sqrt{p^2 + m^2}$

Thus if a π^+ (spin 0, + charged) particle current is identified with the positive energy solution

$$e j^\mu = j^\mu_{em}(\pi^+(p^\mu))$$

$$= 2e|A|^2\left((p^2+m^2)^{1/2}, \underline{p}\right)$$

What can be made of the π^- current

$$e \to e^-$$

$$j^\mu_{em}(\pi^-(p^\mu)) = -2e|A|^2\left((p^2+m^2)^{1/2}, \underline{p}\right)$$

$$= 2e|A|^2\left(-(p^2+m^2)^{1/2}, -\underline{p}\right)$$

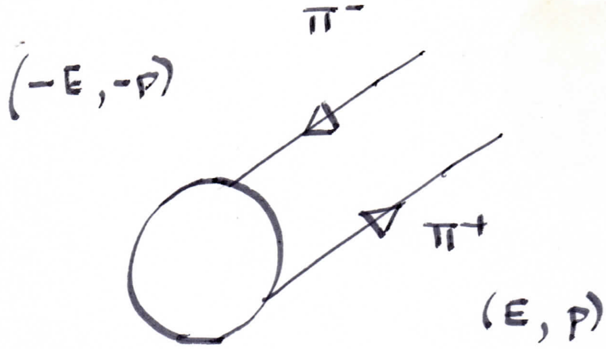
Thus we take the charge sign and apply it to the four vector

Thus

$$\boxed{j^\mu_{em}(\pi^-(p^\mu)) = j^\mu_{em}(\pi^+(-p^\mu))}$$

Thus a π^- current is just a π^+ current with $p^\mu \to -p^\mu$. Thus the negative energy π^+ solutions are interpreted as π^- currents going backwards in time (ie $\underline{p} \to -\underline{p}$).

Thus



These two currents are the same.

Thus the − energy solutions are seen as very useful. $E \to -E$, but also $p \to -p$ gives us the antiparticle current.

> Thus the negative energy states can be interpreted as the antiparticle states travelling backwards in time.