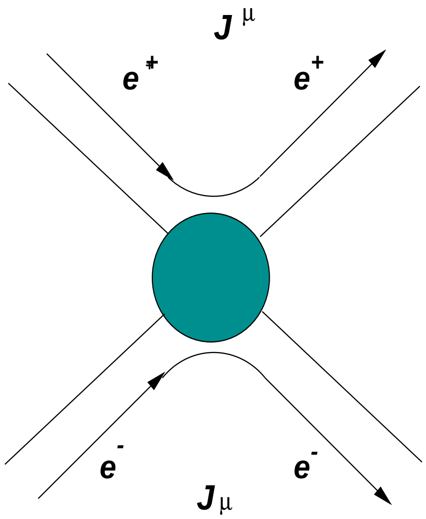
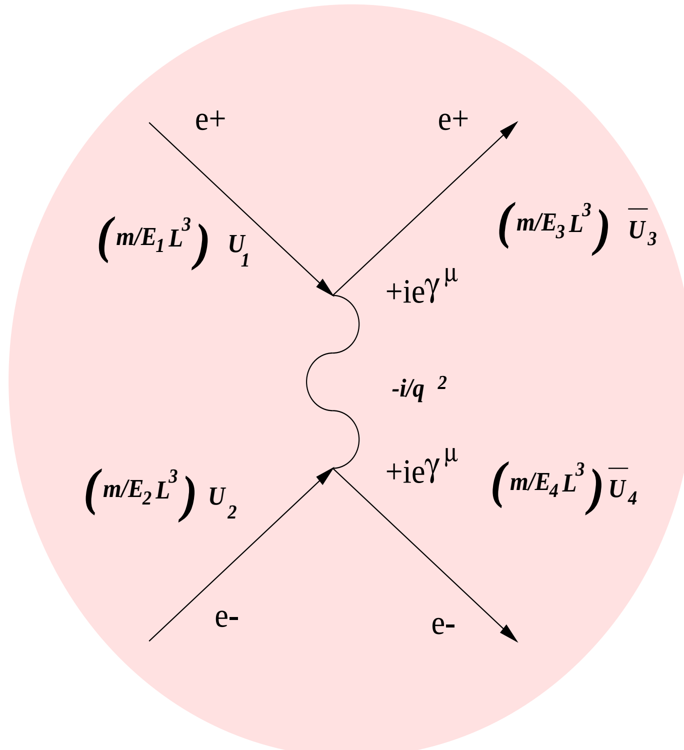
Matrix Element for two spin 1/2 particles to two spin 1/2 particles scattering eg. e+e- scattering

(Postgraduate Relativistic Quatum Mechanics Course: Dr R.Henderson)





The matrix element for the two spin half currents $J^\mu J_\mu$ is

$$V_{if} = \int \overline{\psi_4}\gamma^\mu\psi_2(\frac{e^2}{q^2})\overline{\psi_3}\gamma_\mu\psi_1 \, d\mathbf{x}\, dt$$

We need $V_{if}^\dagger$ in order to calculate the amplitude $V_{if}^\dagger V_{if}$ for this process. Ignoring the integration for the moment we calculate $V_{if}^\dagger$:

$$
\begin{aligned}
V_{if}^\dagger = \widetilde{V_{if}}^* &= \overline{\psi_4}\gamma^\mu\psi_2 \widetilde{\left(\frac{e^2}{q^2}\right)} \overline{\psi_3}\gamma_\mu\psi_1 {}^* \\
&= \widetilde{\overline{\psi_3}\gamma_\mu\psi_1}^* \left(\frac{e^2}{q^2}\right) \widetilde{\overline{\psi_4}\gamma^\mu\psi_2}^* \\
&= \widetilde{\psi_1}^* \gamma_\mu^\dagger \widetilde{\overline{\psi_3}}^* \left(\frac{e^2}{q^2}\right) \widetilde{\psi_2}^* \gamma^{\mu\dagger} \widetilde{\overline{\psi_4}}^* \\
&= \psi_1^\dagger \gamma_\mu^\dagger \gamma_0^\dagger \psi_3 \left(\frac{e^2}{q^2}\right) \psi_2^\dagger \gamma^{\mu\dagger} \gamma_0^\dagger \psi_4 \\
&= \psi_1^\dagger \gamma_\mu^\dagger \gamma_0 \psi_3 \left(\frac{e^2}{q^2}\right) \psi_2^\dagger \gamma^{\mu\dagger} \gamma_0 \psi_4 \\
&= \psi_1^\dagger \gamma_0 \gamma_\mu \gamma_0 \gamma_0 \psi_3 \left(\frac{e^2}{q^2}\right) \psi_2^\dagger \gamma_0 \gamma^\mu \gamma_0 \gamma_0 \psi_4 {}^{1} \\
&= \psi_1^\dagger \gamma_0 \gamma_\mu \psi_3 \left(\frac{e^2}{q^2}\right) \psi_2^\dagger \gamma_0 \gamma^\mu \psi_4 \\
&= \overline{\psi_1} \gamma_\mu \psi_3 \left(\frac{e^2}{q^2}\right) \overline{\psi_2} \gamma^\mu \psi_4
\end{aligned}
$$

Thus we finally have a complete expression for the matrix element squared:

$$
V_{if}^\dagger V_{if} \;=\; (\overline{\psi_1}\gamma_\nu\psi_3 \left(\frac{e^2}{q^2}\right) \overline{\psi_2}\gamma^\nu\psi_4)(\overline{\psi_4}\gamma^\mu\psi_2 \left(\frac{e^2}{q^2}\right) \overline{\psi_3}\gamma_\mu\psi_1)
$$

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[1] since $\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$

The wavefunction consists of a space part and a spinor component $\psi = Ue^{-i(Et-px)}$. If the matrix elements are integrated over the space and time variables the spacial parts of the wavefunction gives the following delta functions

$$\delta(E_3 + E_4 - E_1 - E_2)$$

from the time integral and

$$\delta(\mathbf{P}_3 + \mathbf{P}_4 - \mathbf{P}_1 - \mathbf{P}_2)$$

from the space integral which conserve energy and momentum respectively. This removes the spacial part of the wavefunction and leaves U the spinor part alone. The expression for the matrix element square can then be reexpressed as:

$$V_{if}^{\dagger} V_{if} = (\overline{U_1}\gamma_{\nu} U_3 \left(\frac{e^2}{q^2}\right) \overline{U_2}\gamma^{\nu} U_4)(\overline{U_4}\gamma^{\mu} U_2 \left(\frac{e^2}{q^2}\right) \overline{U_3}\gamma_{\mu} U_1)$$

This form of the matrix element squared would be approriate if we were colliding fully polarised beams and detecting a single polarized state. However in real experiments we are actually colliding unpolarized beams and detecting all final states irrespective of their polarization. This requires that we sum the matrix element over the final states and average over the initial states. Thus we must calculate the following:

$$\frac{1}{4}\sum_{\lambda_1}\sum_{\lambda_2}\sum_{\lambda_3}\sum_{\lambda_4}(\overline{U_1}\gamma_\nu U_3\left(\frac{e^2}{q^2}\right)\overline{U_2}\gamma^\nu U_4)(\overline{U_4}\gamma^\mu U_2\left(\frac{e^2}{q^2}\right)\overline{U_3}\gamma_\mu U_1)$$

Where the sums over λ_n are over the spin states of the nth particle.

$$\frac{e^4}{4q^4}\sum_{\lambda_1}\sum_{\lambda_3}\overline{U_1}\gamma_\nu U_3\underbrace{\sum_{\lambda_2}\overline{U_2}_\alpha\gamma^\nu{}_{\alpha\beta}\overbrace{\sum_{\lambda_4}U_{4\beta}\overline{U_4}_\gamma}\gamma^\mu{}_{\gamma\delta}U_{2\delta}}\overline{U_3}\gamma_\mu U_1$$

The underbraced part of this equation is shown with its explcit summation or contraction variables from α to δ. The overbraced section contains the only dependence on λ_4 so the sum over that variable can be moved to the new position which gives the well known positive energy projection operator which can be written as follows:

$$\sum_{\lambda_4}U_{4\beta}\overline{U_4}_\gamma = \left(\frac{\not{p}_4+m}{2m_4}\right)_{\beta\gamma}$$

The indices $\beta\delta$ remind us that this projection operator is a 4x4 matrix.

$$\frac{e^4}{4q^4}\sum_{\lambda_1}\sum_{\lambda_3}\overline{U_1}\gamma_\nu U_3\underbrace{\sum_{\lambda_2}\overline{U_2}_\alpha\gamma^\nu_{\alpha\beta}\left(\frac{\not{p}_4+m}{2m_4}\right)_{\beta\gamma}\gamma^\mu{}_{\gamma\delta}U_{2\delta}}\overline{U_3}\gamma_\mu U_1$$

The underbraced part of this equation contains the equation's only dependence on λ_2 so the summation sign can be moved as shown below.

$$\sum_{\lambda_2} \overline{U_2}_\alpha \gamma^\nu_{\alpha\beta} \left(\frac{\not{p}_4 + m}{2m_4}\right)_{\beta\gamma} \gamma^\mu_{\gamma\delta} U_{2\delta}$$

This expression can be rearraged as the $U_{2\delta}$ is a single scalar number (not a matrix!) and so can be moved from the end of the equation and placed at the front.

$$\underbrace{\sum_{\lambda_2} U_{2\delta} \overline{U_2}_\alpha}\gamma^\nu_{\alpha\beta} \left(\frac{\not{p}_4 + m}{2m_4}\right)_{\beta\gamma} \gamma^\mu_{\gamma\delta}$$

An equivalent projection operator is then formed for particle 2 by the underbraced section.

$$\sum_{\lambda_2} \overline{U_2}_\delta U_{2\alpha} = \left(\frac{\not{p}_2 + m}{2m_2}\right)_{\delta\alpha}$$

Thus the total underbraced section can now be written as follows:

$$\left(\frac{\not{p}_4 + m}{2m_4}\right)_{\delta\alpha} \gamma^\nu_{\alpha\beta} \left(\frac{\not{p}_2 + m}{2m_2}\right)_{\beta\gamma} \gamma^\mu_{\gamma\delta}$$

This is just a trace of a product of matrices dependent on two indices ν and μ and can be written as the function of particles 2 and 4 as $L(2,4)^{\nu\mu}$:

$$L(2,4)^{\nu\mu} = \frac{1}{4m_2m_4} Tr\left[(\not{p}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma^\mu\right]$$

Thus the original equation for $V^\dagger_{if}V_{if}$ can now be written as

$$\frac{e^4}{4q^4}\frac{1}{4m_2m_4} Tr\left[(\not{p}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma^\mu\right] \underbrace{\sum_{\lambda_1}\sum_{\lambda_3} \overline{U_1}\gamma_\nu U_3 \overline{U_3}\gamma_\mu U_1}$$

By inspection the underbraced part can be replace by a similar traces function $L(1,3)_{\nu\mu}$.

$$L(1,3)_{\nu\mu} \;=\; \frac{1}{4m_1m_3} Tr\left[(\not{p}_1 + m)\gamma_\nu(\not{p}_3 + m)\gamma_\mu\right]$$

Thus finally we can write $V_{if}^\dagger V_{if}$ as the product two traces (requiring $m_2 = m_4 = m_{e^+} = m_2$ for one current $m_1 = m_3 = m_{e^-} = m_1$ for the other in the case of e^+e^- scattering):

$$V_{if}^\dagger V_{if} \;=\; L(2,4)^{\nu\mu} L(1,3)_{\nu\mu}$$

which is:

$$\frac{e^4}{64m_1^2m_2^2q^4} Tr\left[(\not{p}_2 + m_2)\gamma^\nu(\not{p}_4 + m)\gamma^\mu\right] Tr\left[(\not{p}_1 + m_2)\gamma_\nu(\not{p}_3 + m)\gamma_\mu\right]$$

To calculate the matrix element explicitly the traces must be evaluated:

$$\begin{aligned} L(2,4)^{\nu\mu} &= Tr\left[(\not{p}_2 + m)\gamma^\nu(\not{p}_4 + m)\gamma^\mu\right] \\ &= Tr\left[\not{p}_2\gamma^\nu\not{p}_4\gamma^\mu + \not{p}_2\gamma^\nu m_2\gamma^\mu + m_2\gamma^\nu\not{p}_4\gamma^\mu + {m_2}^2\gamma^\nu\gamma^\mu\right] \end{aligned}$$

At this point we need the following trace theorems:

1) $\mathrm{Tr}[\text{odd number of } \gamma\text{'s}] = 0$

2) $\mathrm{Tr}[\gamma^\mu \gamma^\nu] = 4g^{\mu\nu}$

3)

$$\begin{aligned}
Tr[\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu] &= -Tr[\gamma^\beta\gamma^\alpha\gamma^\mu\gamma^\nu] + 2g_{\alpha\beta}Tr[\gamma^\mu\gamma^\nu] \\
-Tr[\gamma^\beta\gamma^\alpha\gamma^\mu\gamma^\nu] &= Tr[\gamma^\beta\gamma^\mu\gamma^\alpha\gamma^\nu] - 2g_{\alpha\mu}Tr[\gamma^\beta\gamma^\nu] \\
Tr[\gamma^\beta\gamma^\mu\gamma^\alpha\gamma^\nu] &= -Tr[\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\alpha] + 2g_{\alpha\nu}Tr[\gamma^\beta\gamma^\mu] \\
-Tr[\gamma^\beta\gamma^\mu\gamma^\nu\gamma^\alpha] &= -Tr[\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu] \\
\Rightarrow 2Tr[\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu] &= \\
2g_{\alpha\beta}Tr[\gamma^\mu\gamma^\nu] - 2g_{\alpha\mu}Tr[\gamma^\beta\gamma^\nu] + 2g_{\alpha\nu}Tr[\gamma^\beta\gamma^\mu] \\
\Rightarrow Tr[\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu] &= 4[g_{\alpha\beta}g_{\mu\nu} - g_{\alpha\mu}g_{\beta\nu} + g_{\alpha\nu}g_{\beta\mu}]
\end{aligned}$$

To match the indices used in the matrix element theorem 3 must be re-expressed as:

$$Tr[\gamma^\alpha\gamma^\nu\gamma^\beta\gamma^\mu] = 4[g_{\alpha\nu}g_{\beta\mu} - g_{\alpha\beta}g_{\nu\mu} + g_{\alpha\mu}g_{\beta\nu}]$$

Thus in the expanded list of traces we can immediately set the terms with two odd products of γ's equal to zero from theorem 1 and the using 3 and 2:

$$
\begin{aligned}
L(2,4)^{\nu\mu} &= Tr\left[\not{p}_2\gamma^\nu\not{p}_4\gamma^\mu + \not{p}_2\gamma^\nu m_2\gamma^\mu + m_2\gamma^\nu\not{p}_4\gamma^\mu + {m_2}^2\gamma^\nu\gamma^\mu\right] \\
&= Tr\left[\not{p}_2\gamma^\nu\not{p}_4\gamma^\mu + {m_2}^2\gamma^\nu\gamma^\mu\right] \\
&= Tr\left[p_{2\alpha}\gamma^\alpha\gamma^\nu p_{4\beta}\gamma^\beta\gamma^\mu + {m_2}^2\gamma^\nu\gamma^\mu\right] \\
&= 4\left[p_{2\alpha}g^{\alpha\nu}p_{4\beta}g^{\beta\mu} - p_{2\alpha}p_{4\beta}g^{\alpha\beta}g^{\nu\mu} + p_{2\alpha}g^{\alpha\mu}p_{4\beta}g^{\beta\nu} + {m_2}^2 g^{\nu\mu}\right] \\
&= 4\left[{p_2}^\nu {p_4}^\mu - p_{2\alpha}{p_4}^\alpha g^{\nu\mu} + {p_2}^\mu {p_4}^\nu + {m_2}^2 g^{\nu\mu}\right]
\end{aligned}
$$

Thus we have the expression from the $L(2,4)_{\nu\mu}$ term

$$
\boxed{L(2,4)^{\nu\mu} = \frac{1}{4m_2^2}\left[{p_2}^\nu {p_4}^\mu - p_2 \cdot p_4 g^{\nu\mu} + {p_2}^\mu {p_4}^\nu + {m_2}^2 g^{\nu\mu}\right]}
$$

So by inspection we have $L(1,3)_{\nu\mu}$:

$$
\boxed{L(1,3)_{\nu\mu} = \frac{1}{4m_1^2}1\left[p_{1_\nu} p_{3_\mu} - p_1 \cdot p_3 g_{\nu\mu} + p_{1_\mu} p_{3_\nu} + {m_1}^2 g_{\nu\mu}\right]}
$$

So it is now possible to write the expression

$$
\begin{aligned}
V_{if}^{\dagger}V_{if} &= \frac{e^4}{4q^4} L(2,4)^{\nu\mu} L(1,3)_{\nu\mu} \\
&= \frac{e^4}{4q^4 m_1^2 m_2^2} \left[{p_2}^{\nu} {p_4}^{\mu} - p_2 \cdot p_4 g^{\nu\mu} + {p_2}^{\mu} {p_4}^{\nu} + {m_2}^2 g^{\nu\mu} \right] \\
&\qquad \left[p_{1\nu} p_{3\mu} - p_1 \cdot p_3 g_{\nu\mu} + p_{1\mu} p_{3\nu} + {m_1}^2 g_{\nu\mu} \right]
\end{aligned}
$$

There are 4x4=16 terms in the result which is easy to see from the following table:

Product	$p_{1\nu}p_{3\mu}$	$-p_1 \cdot p_3 g_{\nu\mu}$	$p_{1\mu}p_{3\nu}$	${m_1}^2 g_{\nu\mu}$
${p_2}^{\nu}{p_4}^{\mu}$	$(p_1 \cdot p_2)(p_4 \cdot p_3)$	$-(p_1 \cdot p_3)(p_2 \cdot p_4)$	$(p_4 \cdot p_1)(p_2 \cdot p_3)$	$m_1^2(p_2 \cdot p_4)$
$-p_2 \cdot p_4 g^{\nu\mu}$	$-(p_2 \cdot p_4)(p_1 \cdot p_3)$	$4(p_2 \cdot p_4)(p_1 \cdot p_3)$	$-(p_2 \cdot p_4)(p_1 \cdot p_3)$	$-4m_1^2(p_2 \cdot p_4)$
${p_2}^{\mu}{p_4}^{\nu}$	$(p_2 \cdot p_3)(p_1 \cdot p_4)$	$-(p_1 \cdot p_3)(p_2 \cdot p_4)$	$(p_1 \cdot p_2)(p_3 \cdot p_4)$	$m_1^2(p_2 \cdot p_4)$
${m_2}^2 g^{\nu\mu}$	$m_2^2(p_1 \cdot p_3)$	$-4m_2^2(p_1 \cdot p_3)$	$m_2^2(p_1 \cdot p_3)$	$4m_1^2 m_2^2$

Since

$$\begin{aligned}
V_{if}^{\dagger}V_{if} &= \frac{e^4}{4q^4m_1^2m_2^2}L(2,4)^{\nu\mu}L(1,3)_{\nu\mu} \\
&= \frac{e^4}{4q^4m_1^2m_2^2}\left[2(p_1\cdot p_2)(p_3\cdot p_4)+2(p_1\cdot p_4)(p_2\cdot p_3)\right. \\
&\quad \left.-2m_1^2(p_2\cdot p_4)-2m_2^2(p_1\cdot p_3)+4m_1^2m_2^2\right] \\
&= \frac{e^4}{2q^4m_1^2m_2^2}\left[(p_1\cdot p_2)(p_3\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3)\right. \\
&\quad \left.-m_1^2(p_2\cdot p_4)-m_2^2(p_1\cdot p_3)+2m_1^2m_2^2\right]
\end{aligned}$$

Thus the full manifestly covariant e^+e^- scattering matrix element (p_1 and p_2 incoming and p_3 and p_4 outgoing e^+ and e^- momenta respectively):

$$\boxed{\begin{aligned}
V_{if}^{\dagger}V_{if} &= \frac{e^4}{2q^4m_1^2m_2^2}\left[(p_1\cdot p_2)(p_3\cdot p_4)+(p_1\cdot p_4)(p_2\cdot p_3)\right. \\
&\quad \left.-m_1^2(p_2\cdot p_4)-m_2^2(p_1\cdot p_3)+2m_1^2m_2^2\right]
\end{aligned}}$$